



# ENGINEERING MATHEMATICS-II

## APPLIED MATHEMATICS

DIPLOMA COURSE IN ENGINEERING  
SECOND SEMESTER

A Publication under  
Government of Tamilnadu  
Distribution of Free Textbook Programme  
(NOT FOR SALE)

Untouchability is a sin  
Untouchability is a crime  
Untouchability is a inhuman

DIRECTORATE OF TECHNICAL EDUCATION  
GOVERNMENT OF TAMILNADU

Government of Tamilnadu  
First Edition – 2015

**Thiru. PRAVEEN KUMAR I.A.S**

Principal Secretary / Commissioner of Technical Education  
Directorate of Technical Education  
Guindy, Chennai- 600025

**Dr. K SUNDARAMOORTHY M.E., Phd.,**

Additional Director of Technical Education (Polytechnics)  
Directorate of Technical Education  
Guindy, Chennai- 600025

**Co-ordinator**

**Er. R.SORNAKUMAR M.E.,**  
Principal  
Dr. Dharmambal Government  
Polytechnic College for Women  
Tharamani, Chennai—113

**Convener**

**Thiru P.L.SANKAR**  
Lecturer (S.G) / Mathematics  
Rajagopal Polytechnic College  
Gudiyatham

**Reviewer**

**Prof. Dr. E. THANDAPANI**  
**UGC EMERITUS FELLOW**  
Ramanujan Institute for Advanced Study in Mathematics  
University of Madras, Chennai 600 005

**Authors**

**Thiru M.RAMALINGAM**  
Lecturer (Sr.Gr) / Mathematics  
Bharathiyar Centenary Memorial  
Polytechnic College  
Ettayapuram

**THIRU.V. SRINIVASAN**  
Lecturer (S.G) / Mathematics  
Arasan Ganesan Polytechnic College  
Sivakasi

**Thirumathi R.S.SUGANTHI**  
Lecturer/Mathematics  
Government Polytechnic College  
Krishnagiri

**Thiru K.NITHYANANDAM**  
HOD/Mathematics  
Adhiparasakthi Polytechnic College  
Melmaruvathur

**Thiru B.R. NARASIMHAN**  
Lecturer (S.G)/Mathematics  
Arulmigu Palaniandavar Polytechnic  
College, Palani.

**Thirumathi N.KANCHANDEVEI**  
Lecturer / Mathematics  
Rajagopal Polytechnic College  
Gudiyatham

This book has been prepared by the Directorate of Technical Education  
This book has been printed on 60 G.S.M Paper

Through the Tamil Nadu Text book and Educational Services Corporation



## FOREWORD

We take great pleasure in presenting this book of mathematics to the students of polytechnic colleges. This book is prepared in accordance with the new syllabus under 'M' scheme framed by the Directorate of Technical Education, Chennai.

This book has been prepared keeping in mind, the aptitude and attitude of the students and modern method of education. The lucid manner in which the concepts are explained, make the teaching and learning process more easy and effective. Each chapter in this book is prepared with strenuous efforts to present the principles of the subject in the most easy to understand and the most easy to workout manner.

Each chapter is presented with an introduction, definitions, theorems, explanation, solved examples and exercises given are for better understanding of concepts and in the exercises, problems have been given in view of enough practice for mastering the concept.

We hope that this book serve the purpose keeping in mind the changing needs of the society to make it lively and vibrating. The language used is very clear and simple which is up to the level of comprehension of students.

We extend our deep sense of gratitude to Thiru. R. Sornakumar Coordinator and Principal, Dr. Dharmambal Government Polytechnic College for women, Chennai and to Thiru P.L. Sankar, Convener and Lecturer / SG, Rajagopal Polytechnic College, Gudiyattam who took sincere efforts in preparing and reviewing this book.

Valuable suggestions and constructive criticisms for improvement of this book will be thankfully acknowledged.

AUTHORS

**30022 ENGINEERING MATHEMATICS – II  
DETAILED SYLLABUS**

**UNIT—I: ANALYTICAL GEOMETRY**

**Chapter - 1.1 EQUATION OF CIRCLE**

**5 Hrs.**

Equation of circle – given centre and radius. General equation of circle – finding centre and radius. Equation of circle on the line joining the points  $(x_1, y_1)$  and  $(x_2, y_2)$  as diameter. Simple Problems.

**Chapter - 1.2 FAMILY OF CIRCLES**

**4 Hrs.**

Concentric circles, contact of two circles (Internal and External) -Simple problems. Orthogonal circles (results only). Problems verifying the condition .

**Chapter - 1.3 INTRODUCTION TO CONIC SECTION**

**5 Hrs.**

Definition of a Conic, Focus, Directrix and Eccentricity. General equation of a conic

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  (statement only). Condition for

conic (i) for circle:  $a = b$  and  $h = 0$  (ii) for pair of straight line:

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(iii) for parabola:  $h^2 - ab = 0$

(iv) for ellipse:  $h^2 - ab < 0$  and (v) for hyperbola:  $h^2 - ab > 0$  . Simple Problems.

**UNIT II: VECTOR ALGEBRA – I**

**Chapter - 2.1 VECTOR - INTRODUCTION**

**5 Hrs.**

Definition of vector - types, addition, and subtraction of Vectors, Properties of addition and subtraction. Position vector. Resolution of vector in two and three dimensions. Direction cosines, Direction ratios. Simple problems.

**Chapter - 2.2 SCALAR PRODUCT OF VECTORS**

**5 Hrs.**

Definition of Scalar product of two vectors – Properties – Angle between two vectors. Simple Problems.

**Chapter - 2.3 APPLICATION OF SCALAR PRODUCT**

**4 Hrs.**

Geometrical meaning of scalar product. Work done by Force. Simple Problems.

**UNIT III: VECTOR ALGEBRA – II**

**Chapter - 3.1 VECTOR PRODUCT OF TWO VECTORS**

**5 Hrs.**

Definition of vector product of two vectors. Geometrical meaning. Properties – Angle between two vectors – unit vector perpendicular to two vectors. Simple Problems.

**Chapter - 3.2 APPLICATION OF VECTOR PRODUCT OF TWO VECTORS & SCALAR TRIPLE PRODUCT**

**5 Hrs.**

Definition of moment of a force. Definition of scalar product of three vectors – Geometrical meaning – Coplanar vectors. Simple Problems.

**Chapter - 3.3 VECTOR TRIPLE PRODUCT & PRODUCT OF MORE VECTORS** **4 Hrs.**

Definition of Vector Triple product, Scalar and Vector product of four vectors Simple Problems.

**UNIT IV: INTEGRAL CALCULUS – I**

**Chapter - 4.1 INTEGRATION – DECOMPOSITION METHOD** **5 Hrs.**

Introduction - Definition of integration – Integral values using reverse process of differentiation – Integration using decomposition method. Simple Problems.

**Chapter - 4.2 INTEGRATION BY SUBSTITUTION** **5 Hrs.**

Integrals of the form  $\int [f(x)]^n f'(x) dx, n \neq -1$ ,  $\int \frac{f'(x)}{f(x)} dx$  and  $\int F[f(x)] f'(x) dx$ . Simple Problems.

**Chapter - 4.3 STANDARD INTEGRALS** **4 Hrs.**

Integrals of the form  $\int \frac{dx}{a^2 \pm x^2}$ ,  $\int \frac{dx}{x^2 - a^2}$  and  $\int \frac{dx}{\sqrt{a^2 - x^2}}$ . Simple Problems

**UNIT V: INTEGRAL CALCULUS – II**

**Chapter - 5.1 INTEGRATION BY PARTS** **5 Hrs.**

Integrals of the form  $\int x \sin nx dx$ ,  $\int x \cos nx dx$ ,  $\int x e^{nx} dx$ ,  $\int x^n \log x dx$  and

**Chapter - 5.2 BERNOULLI'S FORMULA** **4 Hrs.**

Evaluation of the integrals  $\int x^m \sin nx dx$ ,  $\int x^m \cos nx dx$  and  $\int x^m e^{nx} dx$  where  $m \leq 2$  using Bernoulli's formula. Simple Problems.

**Chapter - 5.3 DEFINITE INTEGRALS** **5 Hrs.**

Definition of definite Integral. Properties of definite Integrals - Simple Problems.

**30023 APPLIED MATHEMATICS  
DETAILED SYLLABUS**

**UNIT—I: PROBABILITY DISTRIBUTION – I**

**Chapter - 1.1 RANDOM VARIABLE** **5Hrs.**

Definition of Random variable – Types – Probability mass function –Probability density function. Simple Problems.

**Chapter - 1.2 MATHEMATICAL EXPECTATION** **4Hrs.**

Mathematical Expectation of discrete random variable, mean and variance. Simple Problems.

**Chapter - 1.3 BINOMIAL DISTRIBUTION** **5Hrs.**

Definition of Binomial distribution  $P(X = x) = {}^n C_x p^x q^{n-x}$  where

$x = 0, 1, 2, \dots$  Statement only. Expression for mean and variance. Simple Problems.

**UNIT—II: PROBABILITY DISTRIBUTION – II**

**Chapter - 2.1 POISSON DISTRIBUTION** **5 Hrs.**

Definition of Poission distribution  $P(X = x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$  where  $x = 0, 1, 2, \dots$

(statement only). Expressions of mean and variance. Simple Problems.

**Chapter - 2.2 NORMAL DISTRIBUTION** **5 Hrs.**

Definition of normal and standard normal distribution – statement only. Constants of normal distribution (Results only). Properties of normal distribution - Simple problems using the table of standard normal distribution.

**Chapter - 2.3 CURVE FITTING** **4 Hrs.**

Fitting of straight line using least square method (Results only). Simple problems.

**UNIT III : APPLICATION OF DIFFERENTIATION**

**Chapter – 3.1 VELOCITY AND ACCELERATION** **5 Hrs.**

Velocity and Acceleration – Simple Problems.

**Chapter - 3.2 TANGENT AND NORMAL** **4 Hrs.**

Tangent and Normal – Simple Problems.

**Chapter - 3.3 MAXIMA AND MINIMA** **5 Hrs.**

Definition of increasing and decreasing functions and turning points. Maxima and Minima of single variable only – Simple Problems.

**UNIT IV: APPLICATION OF INTEGRATION – I**

**Chapter - 4.1 AREA AND VOLUME** **5 Hrs.**

Area and Volume – Area of Circle. Volume of Sphere and Cone – Simple Problems.

**Chapter - 4.2 FIRST ORDER DIFFERENTIAL EQUATION** **5 Hrs.**

Solution of first order variable separable type differential equation .Simple Problems.

**Chapter - 4.3 LINEAR TYPE DIFFERENTIAL EQUATION** **4 Hrs.**

Solution of linear differential equation. Simple problems.

## UNIT V : APPLICATION OF INTEGRATION – II

### Chapter – 5.1 SECOND ORDER DIFFERENTIAL EQUATION – I

4 Hrs.

Solution of second order differential equation with constant co-efficients in the form

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

where  $a, b$  and  $c$  are constants. Simple Problems.

### Chapter - 5.2 SECOND ORDER DIFFERENTIAL EQUATION – II

5 Hrs.

Solution of second order differential equations with constant co- efficients in the form

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$$

where  $a, b$  and  $c$  are constants and

$$f(x) = k e^{mx}$$

. Simple Problems.

### Chapter - 5.3 SECOND ORDER DIFFERENTIAL EQUATION – III

5 Hrs.

Solution of second order differential equation with constant co-efficients in the form

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$$

where  $a, b$  and  $c$  are constants and

$$f(x) = k \sin mx \text{ or } k \cos mx$$

. Simple Problems.

# *NOTES*

# UNIT – I

## ANALYTICAL GEOMETRY

### 1.1 Equation of Circle

Equation of circle given centre and radius. General equation of circle finding centre and radius. Equation of circle on the line joining the points  $(x_1, y_1)$  and  $(x_2, y_2)$  as diameter. Simple problems.

### 1.2 Family of Circles:

Concentric circles, contact of two circles (internal and external). Simple problems, Orthogonal circles (results only), problems verifying the condition.

### 1.3 Introduction to conic section:

Definition of a conic, Focus, Directrix and Eccentricity, General equation of a conic  $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$  (Statement only). Condition for conic (i) for circle :  $a = b$  and  $h = 0$ .

(ii) for pair of straight line  $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(iii) for parabola :  $h^2 - ab = 0$  (iv) for ellipse :  $h^2 - ab < 0$

(v) for hyperbola :  $h^2 - ab > 0$  – Simple problems.

## 1.1 EQUATION OF CIRCLE

### Definition:

A circle is the locus of a point which moves in a plane such that its distance from a fixed point is constant. The fixed point is the centre of the circle and the constant distance is the radius of the circle.

### 1.1.1 Equation of the circle with centre $(h, k)$ and radius ' $r$ ' units:

Given: The centre and radius of the circle are  $(h, k)$  and ' $r$ ' units. Let  $P(x, y)$  be any point on the circle.

From Fig.(1.1)

$$CP = r$$

$$\sqrt{(x-h)^2 + (y-k)^2} = r \quad \text{(Using distance formula)}$$

$$(x-h)^2 + (y-k)^2 = r^2 \quad \text{———— (1)}$$

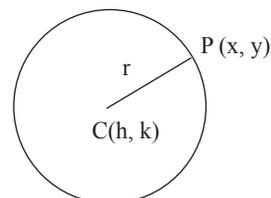


Fig.(1.1)

### Note:

The equation of the circle with centre  $(0, 0)$  and radius ' $r$ ' units is  $x^2 + y^2 = r^2$ .

### 1.1.2 General Equation of the circle:

The General Equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{———— (2)}$$

Add  $g^2, f^2$  on both sides

$$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c$$

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

$$[x - (-g)]^2 + [y - (-f)]^2 = \left[ \sqrt{g^2 + f^2 - c} \right]^2 \quad \text{.....(3)}$$

Equation (3) is of the form equation (1)

The equation (2) represents a circle with centre  $(-g, -f)$  and radius  $\sqrt{g^2 + f^2 - c}$ .

**Note:**

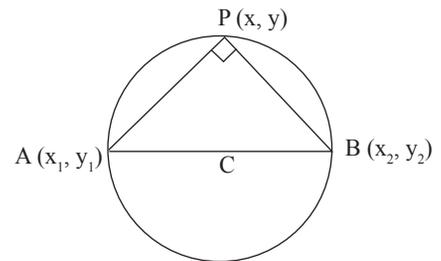
1. In the equation of the circle co-efficient of  $x^2 =$  co-efficient of  $y^2$ .

2. Centre of the circle =  $\left[ -\frac{1}{2} \text{co-efficient of } x, -\frac{1}{2} \text{co-efficient of } y \right]$

3. radius of the circle  $\sqrt{g^2 + f^2 - c}$ .

### 1.1.3 Equation of the circle on the line joining the points $(x_1, y_1)$ and $(x_2, y_2)$ as diameter.

Let  $A(x_1, y_1)$  and  $B(x_2, y_2)$  be the given end points of a diameter. Let  $P(x, y)$  be any point on the circle.



$\therefore$  Angle in a semi-circle is  $90^\circ$

$\therefore \angle APB = 90^\circ$

$\Rightarrow AP \perp BP$

(Slope of AP) (Slope of BP) =  $-1$ .

$$\left( \frac{y - y_1}{x - x_1} \right) \left( \frac{y - y_2}{x - x_2} \right) = -1$$

$$(y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$$

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

This is the Diameter form of the equation of a circle.

### PART – A

1. Find the equation of the circle whose centre is  $(-1, 2)$  and radius 5 units.

**Solution:**

Equation of the circle is

$$(x - h)^2 + (y - k)^2 = r^2 \quad \left| \begin{array}{l} (-1, 2), r = 5 \\ (h, K) \end{array} \right.$$

$$(x + 1)^2 + (y - 2)^2 = 5^2 \quad \left| \begin{array}{l} (h, K) \end{array} \right.$$

$$x^2 + 2x + 1 + y^2 - 4y + 4 = 25$$

$$x^2 + y^2 + 2x - 4y - 20 = 0$$

2. Find the centre and radius of the circle

$$x^2 + y^2 - 4x + 8y - 7 = 0$$

**Solution:**

$$\begin{aligned} \text{Centre} = C(-g, -f) & \left| \begin{array}{l} 2g = -4, 2f = 8, c = -7 \\ g = -2 \quad f = 4 \end{array} \right. \\ & = C(2, -4) \end{aligned}$$

$$\begin{aligned} r &= \sqrt{g^2 + f^2 - c} \\ &= \sqrt{(-2)^2 + (4)^2 - (-7)} \\ &= \sqrt{4 + 16 + 7} \\ &= \sqrt{27} \end{aligned}$$

3. Obtain the equation of the circle on the line joining  $(-1, 2)$  and  $(-3, 5)$  as diameter.

**Solution:**

Equation of the circle is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$(x + 1)(x + 3) + (y - 2)(y - 5) = 0$$

$$x^2 + 4x + 3 + y^2 - 7y + 10 = 0$$

$$x^2 + y^2 + 4x - 7y + 13 = 0$$

### PART – B

1. Find the equation of the circle passing through  $(2, 1)$  and having its centre at  $(-3, -4)$ .

**Solution:**

$$\begin{aligned} r &= \sqrt{(-3 - 2)^2 + (-4 - 1)^2} \\ &= \sqrt{25 + 25} \\ &= \sqrt{50} \end{aligned}$$

Equation of the circle  $(x - h)^2 + (y - k)^2 = r^2$

$$(x + 3)^2 + (y + 4)^2 = (\sqrt{50})^2$$

$$x^2 + 6x + 9 + y^2 + 8y + 16 = 50$$

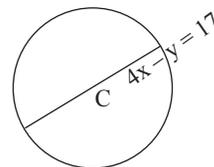
$$x^2 + y^2 + 6x + 8y - 25 = 0$$

2. Show that the line  $4x - y = 17$  is the diameter of the circle  $x^2 + y^2 - 8x + 2y + 3 = 0$ .

**Solution:**

$$x^2 + y^2 - 8x + 2y + 3 = 0$$

$$\begin{aligned} \text{Centre} = C(-g, -f) & \left| \begin{array}{l} 2g = -8, 2f = 2 \\ g = -4, f = 1 \end{array} \right. \\ & = C(4, -1) \end{aligned}$$



Put  $x = 4, y = -1$  in

$$4x - y = 17$$

$$4(4) - (-1) = 17$$

$$16 + 1 = 17$$

$$17 = 17$$

$\therefore (4, -1)$  lies on the line  $4x - y = 17$ .

$\therefore 4x - y = 17$  is the diameter of the circle.

3. Find the centre and radius of the circle  $4x^2 + 4y^2 - 8x + 16y + 19 = 0$ .

**Solution:**

$$4x^2 + 4y^2 - 8x + 16y + 19 = 0$$

Divide by 4

$$x^2 + y^2 - 2x + 4y + \frac{19}{4} = 0$$

$$\begin{aligned} \text{Centre} = C(-g, -f) & \left| \begin{array}{l} 2g = -2, \quad 2f = 4, \quad c = \frac{19}{4} \\ g = -1, \quad f = 2 \end{array} \right. \\ & = C(1, -2) \end{aligned}$$

$$r = \sqrt{g^2 + f^2 - c} = \sqrt{(-1)^2 + (2)^2 - \frac{19}{4}} = \sqrt{1 + 4 - \frac{19}{4}} = \sqrt{\frac{20 - 19}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

### PART - C

1. Find the equation of the circle, two of whose diameters are  $x + y = 6$  and  $x + 2y = 4$  and whose radius is 10 units.

**Solution:**

$$x + y = 6 \quad \text{--- (1)}$$

$$x + 2y = 4 \quad \text{--- (2)}$$

$$(1) - (2) \Rightarrow -y = 2 \quad \therefore y = -2$$

Substitute  $y = -2$  in (1)

$$x + (-2) = 6$$

$$x = 6 + 2$$

$$x = 8$$

$$\text{Centre} = C(8, -2)$$

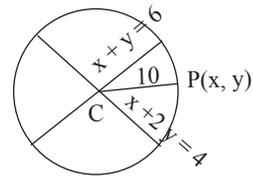
$$\text{Equation of the circle} \quad h, k = 8, -2$$

$$(x - h)^2 + (y - k)^2 = r^2 \quad r = 10$$

$$(x - 8)^2 + (y + 2)^2 = 10^2$$

$$x^2 - 16x + 64 + y^2 + 4y + 4 = 100$$

$$x^2 + y^2 - 16x + 4y - 32 = 0$$



2. Show that the point  $(8, 9)$  lies on the circle  $x^2 + y^2 - 10x - 12y + 43 = 0$ . Find the co-ordinates of the other end of the diameter of the circle through this point.

**Solution:**

Put  $x = 8, y = 9$  in

$$x^2 + y^2 - 10x - 12y + 43 = 0$$

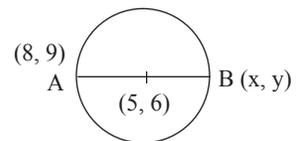
$$(8)^2 + (9)^2 - 10(8) - 12(9) + 43 = 0$$

$$64 + 81 - 80 - 108 + 43 = 0$$

$$188 - 188 = 0$$

$$0 = 0$$

$(8, 9)$  lies on the circle  $x^2 + y^2 - 10x - 12y + 43 = 0$ .



$$\begin{array}{l} \text{Centre} = C(-g, -f) \\ = C(5, 6) \end{array} \left| \begin{array}{l} 2g = -10 \\ g = -5 \\ 2f = -12 \\ f = -6 \end{array} \right.$$

Centre C is the mid point of diameter AB.

$$\therefore (5, 6) = \left[ \frac{8+x}{2}, \frac{9+y}{2} \right]$$

$$\text{i.e. } 5 = \frac{8+x}{2}, \quad 6 = \frac{9+y}{2}$$

$$10 = 8 + x, \quad 12 = 9 + y$$

$$x = 2, \quad y = 3$$

$\therefore$  The other end of the diameter is (2, 3).

**Note:**

(i) If  $(x_1, y_1)$  lies on the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  then  $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$ .

(ii) If  $(x_1, y_1)$  lies outside the circle  $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$ .

(iii) If  $(x_1, y_1)$  lies inside the circle, then  $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$

## 1.2 FAMILY OF CIRCLES

### 1.2.1 Concentric Circles:

Two or more circles having the same centre but differ in radii are called concentric circles.

For examples:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$x^2 + y^2 + 2gx + 2fy + p = 0$$

$$x^2 + y^2 + 2gx + 2fy + q = 0$$

are concentric circles which have same centre  $(-g, -f)$ .

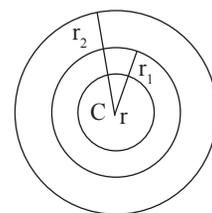


Fig.1.2.1

Note: Equation of concentric circles differ only by the constant term.

### 1.2.2 Contact of circles:

**Case (i)**

Two circles touch externally if the distance between their centres is equal to sum of their radii.

i.e.  $c_1c_2 = r_1 + r_2$ .

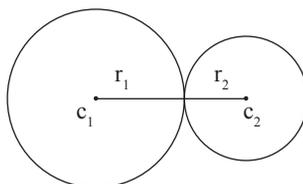


Fig.1.2.2

**Case (ii)**

Two circles touch internally if the distance between their centres is equal to difference of their radii.

$$\text{i.e } c_1 c_2 = r_1 - r_2 \text{ or } r_2 - r_1$$

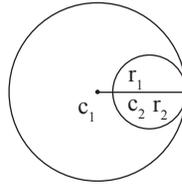


Fig.1.2.3

**1.2.3 Orthogonal Circles:**

Two circles are said to be orthogonal if the tangents at their point of intersection are perpendicular to each other.

**Condition for two circles to cut orthogonally.** (Results only).

Let the equation of the circles be

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

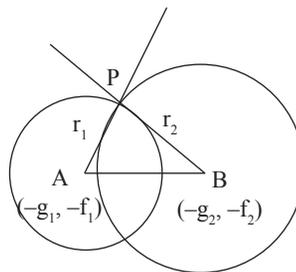


Fig.1.2.4

Let the circles cut each other at the point P.

The centres and radii of the circles are  $A_1(-g_1, -f_1)$ ,  $B(-g_2, -f_2)$

$$r_1 = AP = \sqrt{g_1^2 + f_1^2 - c_1}, \quad r_2 = PB = \sqrt{g_2^2 + f_2^2 - c_2}.$$

From fig.(1 2 4) APB is a right angled triangle.

$$AB^2 = AP^2 + PB^2$$

$$(-g_1 + g_2)^2 + (-f_1 + f_2)^2 = g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2.$$

$$g_1^2 + g_2^2 - 2g_1g_2 + f_1^2 + f_2^2 - 2f_1f_2 = g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2$$

$$-2g_1g_2 - 2f_1f_2 = -c_1 - c_2$$

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

This is the required condition for two circles to cut orthogonally.

**Note:**

When the centre of any one circle is at the origin then condition for orthogonal circles is  $c_1 + c_2 = 0$ .

**PART – A**

1. Find whether the circles  $x^2 + y^2 - 4x + 6y + 8 = 0$  and  $x^2 + y^2 - 4x + 6y - 8 = 0$  are concentric.

**Solution:**

From the two given equations of the circles we observe that the constant term alone differs i.e with the same centre.  $(2, -3)$ .

$\therefore$  The given circles are concentric circles.

2. Find the distance between the centres of the circles  $x^2 + y^2 - 4x - 6y + 9 = 0$  and  $x^2 + y^2 + 2x + 2y - 7 = 0$ .

**Solution:**

$$x^2 + y^2 - 4x - 6y + 9 = 0 \text{ and } x^2 + y^2 + 2x + 2y - 7 = 0.$$

$$\begin{array}{l} 2g_1 = -4, 2f_1 = -6 \\ g_1 = -2, f_1 = -3 \end{array} \quad \left| \quad \begin{array}{l} 2g_2 = 2, 2f_2 = 2 \\ g_2 = 1, f_2 = 1 \end{array} \right.$$

$$\text{Centres: } c_1(2, 3) \quad c_2(-1, -1)$$

$$\begin{aligned} \text{Distance } c_1c_2 &= \sqrt{(2+1)^2 + (3+1)^2} \\ &= \sqrt{9+16} \\ &= 5 \end{aligned}$$

3. Find the equation of the circle concentric with the circle  $x^2 + y^2 + 5 = 0$  and passing through  $(1, 0)$ .

**Solution:**

Equation of the circle concentric with

$$x^2 + y^2 + 5 = 0 \text{ is}$$

$$x^2 + y^2 + k = 0$$

Put  $x = 1$  and  $y = 0$  in  $x^2 + y^2 + k = 0$

$$(1)^2 + 0 + k = 0$$

$$k = -1$$

$$\therefore x^2 + y^2 - 1 = 0$$

which is the required equation of the circle.

4. Verify whether the circles  $x^2 + y^2 + 10 = 0$  and  $x^2 + y^2 - 10 = 0$  cut orthogonally.

**Solution:**

When the centre of any one of the circle is at the origin then

$$\text{Condition for orthogonality } c_1 + c_2 = 0$$

$$\text{i.e } 10 - 10 = 0$$

$$0 = 0$$

$\therefore$  The circles cut orthogonally.

**PART – B**

1. Find the equation of the circle which is concentric with the circle  $x^2 + y^2 - 8x + 12y + 15 = 0$  and passes through  $(5, 4)$ .

**Solution:**

Equation of the concentric circle be

$$x^2 + y^2 - 8x + 12y + k = 0 \quad \text{————— (1)}$$

Put  $x = 5, y = -4$  in (1)

$$x^2 + y^2 - 8x + 12y - 29 = 0$$

$$(5)^2 + (4)^2 - 8(5) + 12(4) + k = 0$$

$$25 + 16 - 40 + 48 + k = 0$$

$$29 + k = 0$$

$$k = -49$$

$\therefore$  The required equation of the circle is  $x^2 + y^2 - 8x + 12y - 49 = 0$ .

2. Find the equation of the circle concentric with the circle  $x^2 + y^2 + 3x - 7y + 1 = 0$  and having radius 5 units.

**Solution:**

Centre of the circle  $x^2 + y^2 + 3x - 7y + 1 = 0$  is  $\left(\frac{-3}{2}, \frac{7}{2}\right)$

$\therefore$  Centre of the concentric circle is  $\left(\frac{-3}{2}, \frac{7}{2}\right)$  and radius 5 units.

Equation of the circle

$$(x - h)^2 + (y - k)^2 = r^2 \quad \left| \begin{array}{l} \left(\frac{-3}{2}, \frac{7}{2}\right), r = 5 \\ (4, k) \end{array} \right.$$

$$\left(x + \frac{3}{2}\right)^2 + \left(y - \frac{7}{2}\right)^2 = 5^2$$

$$x^2 + \frac{9}{4} + 3x + y^2 - 7y + \frac{49}{4} = 25$$

$$4x^2 + 4y^2 + 12x - 28y - 42 = 0$$

3. Show that the circles  $x^2 + y^2 - 8x + 6y - 23 = 0$  and  $x^2 + y^2 - 2x - 5y + 16 = 0$  are orthogonal.

**Solution:**

$$x^2 + y^2 - 8x + 6y - 23 = 0 \text{ and } x^2 + y^2 - 2x - 5y + 16 = 0$$

$$g_1 = -4, f_1 = 3, c_1 = -23, \quad g_2 = -1, f_2 = -5/2, c_2 = 16$$

Condition for orthogonality

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

$$2(-4)(-1) + 2(3)\left(-\frac{5}{2}\right) = -23 + 16$$

$$8 - 15 = -7$$

$$-7 = -7$$

$\therefore$  The circles cut orthogonally.

### PART - C

1. Show that the circles  $x^2 + y^2 - 4x + 6y + 8 = 0$  and  $x^2 + y^2 - 10x - 6y + 14 = 0$  touch each other.

**Solution:**

$$x^2 + y^2 - 4x + 6y + 8 = 0 \text{ and } x^2 + y^2 - 10x - 6y + 14 = 0$$

$$c_1 (2, -3)$$

$$c_2 (5, 3)$$

$$r_2 = \sqrt{(-5)^2 + (-3)^2 - 14}$$

$$r_1 = \sqrt{(-2)^2 + (3)^2 - 8}$$

$$= \sqrt{20}$$

$$= \sqrt{5}$$

$$= 2\sqrt{5}$$

$$\begin{aligned}
 c_1 c_2 &= \sqrt{(2-5)^2 + (-3-3)^2} \\
 &= \sqrt{9+36} \\
 &= \sqrt{45} \\
 &= 3\sqrt{5} \\
 \therefore c_1 c_2 &= r_1 + r_2
 \end{aligned}$$

$\therefore$  the circles touch each other externally.

2. Show that the circles  $x^2 + y^2 - 2x + 6y + 6 = 0$  and  $x^2 + y^2 - 5x + 6y + 15 = 0$  touch each other.

**Solution:**

$$\begin{array}{l}
 x^2 + y^2 - 2x + 6y + 6 = 0 \\
 c_1(1, -3) \\
 r_1 = \sqrt{(-1)^2 + (-3)^2 - 6} \\
 = \sqrt{1+9-6} \\
 = \sqrt{4} \\
 r_1 = 2
 \end{array}
 \left|
 \begin{array}{l}
 x^2 + y^2 - 5x + 6y + 15 = 0 \\
 c_2 = \left(\frac{5}{2}, -3\right) \\
 r_2 = \sqrt{\frac{25}{4} + 9 - 15} \\
 = \sqrt{\frac{25}{4} - 6} \\
 r_2 = \frac{1}{2}
 \end{array}
 \right.
 \begin{array}{l}
 c_1 c_2 = \sqrt{\left(\frac{5}{2} - 1\right)^2 + (-3 + 3)^2} \\
 = \sqrt{\left(\frac{3}{2}\right)^2} \\
 c_1 c_2 = \frac{3}{2} \\
 r_1 - r_2 = 2 - \frac{1}{2} \\
 = \frac{4-1}{2} = \frac{3}{2} \\
 = c_1 c_2
 \end{array}$$

$\therefore$  the circles touch each other internally.

## 1.3 INTRODUCTION TO CONIC SECTION

### 1.3.1 Definitions

#### Conic

A conic is defined as the locus of a point which moves such that its distance from a fixed point is always 'e' times its distance from a fixed straight line.

#### Focus:

The fixed point is called the focus of the conic.

#### Directrix:

The fixed straight line is called the directrix of the conic.

#### Eccentricity:

The constant ratio is called the eccentricity of the conic.

From Fig.(1.3)

S is the focus line XM is the Directrix 'e' eccentricity of the conic then<sup>X</sup>

$$\frac{SP}{PM} = e$$

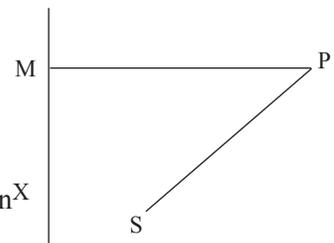


Fig.(1.3)

This is known as Focus-Directrix property.

**Note:**

- (i) If  $e < 1$ , the conic is called an ellipse.
- (ii) If  $e = 1$ , the conic is called a parabola.
- (iii) If  $e > 1$ , the conic is called a hyperbola.

### 1.3.2 General equation of a Conic:

Let the focus be S ( $x_1, y_1$ ) and the directrix be the line  $ax + by + c = 0$ .

P ( $x, y$ ) be any point on it From Fig.(1.3)

$$SP = \sqrt{(x - x_1)^2 + (y - y_1)^2}$$

Perpendicular distance of P from  $ax + by + c = 0$  is

$$\pm \frac{ax + by + c}{\sqrt{a^2 + b^2}}$$

Always,  $\frac{SP}{PM} = e$  (eccentricity of the conic)

$$\frac{\sqrt{(x - x_1)^2 + (y - y_1)^2}}{\pm \frac{ax + by + c}{\sqrt{a^2 + b^2}}} = e$$

Squaring on both sides

$$\left[ \sqrt{(x - x_1)^2 + (y - y_1)^2} \right]^2 = e^2 \left[ \frac{(ax + by + c)^2}{a^2 + b^2} \right]$$

on simplification we get an equation of the second degree of the form.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

∴ The equation of any conic is of second degree in x and y.

**Note:**

Any equation of second degree in x and y represents any one of the following curves.

The general form of equation of the second degree (or) General equation of a conic.

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  represents

(i) a circle if  $a = b$  and  $h = 0$ .

(ii) a pair of straight line if  $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ .

$$\text{(or) } abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

(iii) a parabola if  $h^2 = ab$ .

(iv) an ellipse if  $h^2 < ab$ .

(v) a hyperbola if  $h^2 > ab$ .

## PART – A

1. Show that  $x^2 - 4xy + 4y^2 - 24x - 12y + 24 = 0$  represents a parabola.

**Solution:**

Condition for a second degree equation to represent a parabola is  $h^2 = ab$ .

$$\begin{array}{l|l} h^2 = ab & 2h = -4, \quad a = c, \quad b = 4 \\ (-2)^2 = (1)(4) & h = -2 \\ 4 = 4 & \end{array}$$

$\therefore$  The given equation represents a parabola.

2. Show that the  $x^2 + y^2 - 4xy + 4x - 4 = 0$  represents a hyperbola.

**Solution:**

Condition for General equation of a conic to represent a hyperbola is  $h^2 - ab > 0$ .

$$\begin{array}{l|l} h^2 = ab & a = b = 1, \quad 2h = -4 \\ = (-2)^2 - (1)(1) & h = -2 \\ = 4 - 1 & \\ = 3 > 0 & \end{array}$$

$\therefore$  The given equation represents a parabola.

3. Show that  $5x^2 + 5y^2 + 2x + y + 1 = 0$  represents a circle.

**Solution:**

Condition for conic to represent a circle is

$$a = b \text{ and } h = 0$$

$$a = b = 5 \text{ and } h = 0$$

$\therefore$  The given equation represents a circle.

## PART – B

1. Find the equation of the parabola whose focus is the point (2, 1) and whose directrix is the straight line  $2x + y + 1 = 0$ .

**Solution:**

For parabola  $e = 1$ .

Given: Focus is S (2, 1) and directrix is  $2x + y + 1 = 0$ .

$$\text{Always } \frac{SP}{PM} = e = 1$$

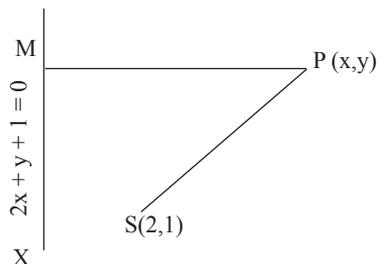
$$\frac{\sqrt{(x-2)^2 + (y-1)^2}}{\pm \frac{2x+y+1}{\sqrt{(2)^2 + (1)^2}}} = 1$$

$$\sqrt{(x-2)^2 + (y-1)^2} = \pm \frac{2x+y+1}{\sqrt{5}}$$

$$(x-2)^2 + (y-1)^2 = \frac{(2x+y+1)^2}{5}$$

$$= [x^2 - 4x + 4 + y^2 - 2y + 1] = 4x^2 + y^2 + 1 + 4xy + 2y + 4x$$

$$5x^2 - 4xy + 4y^2 - 24x - 12y + 24 = 0$$



2. Find the equation of the hyperbola whose eccentricity is  $\sqrt{3}$ , focus is (1, 2) and directrix  $2x + y = 1$ .

**Solution:**

$$\text{Always } \frac{SP}{PM} = e:$$

Given  $e = \sqrt{3}$  focus S (1, 2) directrix  $2x + y = 1$ . Let P (x, y) be any point on the hyperbola.

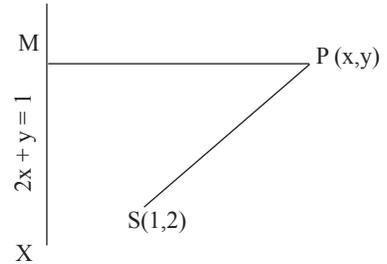
$$SP^2 = e^2 PM^2$$

$$(x-1)^2 + (y-2)^2 = 3 \left[ \frac{2x+y-1}{\sqrt{(2)^2 + (1)^2}} \right]^2$$

$$x^2 - 2x + 1 + y^2 - 4y + 4 = \frac{3(2x+y-1)^2}{5}$$

$$5(x^2 + y^2 - 2x - 4y + 5) = 3(4x^2 + y^2 + 1 + 4xy - 2y - 4x)$$

$$7x^2 + 12xy - 2y^2 - 2x + 14y - 22 = 0$$



### PART – C

1. Show that the equation  $12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$  represents a pair of st.lines.

**Solution:**

Condition for a second degree equation to represent a pair of straight line.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \text{ (or) } \begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$$

$$a = 12, \quad 2h = 7, \quad b = -10, \quad 2g = 13, \quad 2f = 45 \quad c = -35$$

$$2a = 24 \quad 2b = -20$$

$$\begin{vmatrix} 24 & 7 & 13 \\ 7 & -20 & 45 \\ 13 & 45 & -70 \end{vmatrix}$$

$$= 24(1400 - 2025) - 7(-490 - 585) + 13(315 + 260)$$

$$= 24(-625) - 7(-1075) + 13(575)$$

$$= -15000 + 7525 + 7475$$

$$= 0$$

∴ The given equation represents a pair of straight line.

2. Find the value of 'k' when the equation  $3x^2 + 7xy + 2y^2 + 5x + 5y + k = 0$  may represent a pair of straight line.

**Solution:**

Let the given equation represent a pair of straight line.

Condition:

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \text{ (or) } \begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

$$3x^2 + 7xy + 2y^2 + 5x + 5y + k = 0$$

$$a = 3 \quad 2h = 7 \quad b = 2 \quad 2g = 5 \quad 2f = 5 \quad c = k$$

$$2a = 6 \quad 2b = 4 \quad 2c = 2k$$

$$\begin{vmatrix} 6 & 7 & 5 \\ 7 & 4 & 5 \\ 5 & 5 & 2k \end{vmatrix} = 0$$

$$6(8k - 25) - 7(14k - 25) + 5(35 - 20) = 0$$

$$48k - 150 - 98k + 175 + 75 = 0$$

$$-50k + 100 = 0$$

$$-50k = -100$$

$$k = 2$$

**EXERCISE**

**PART - A**

- Find the equation of the circle whose centre and radius are given as  
(i) (3, 2) ; 4 units    (ii) (-5, 7), 3 units    (iii) (-5, -4) ; 5 units    (iv) (6, -2), 10 units
- Find the centre and radius of the circle  $(x + 5)^2 + (y - 2)^2 = 7$ .
- Find the centre and radius of the following circles:  
(i)  $x^2 + y^2 - 12x - 8y + 2 = 0$     (ii)  $x^2 + y^2 + 7x + 5y - 1 = 0$   
(iii)  $2x^2 + 2y^2 - 6x + 12y - 4 = 0$     (iv)  $x^2 + y^2 = 100$
- Find the equation of the circle described on line joining the following points as diameter.  
(i) (3, 5) and (2, 7)    (ii) (-1, 0) and (0, -3)  
(iii) (0, 0) and (4, 4)    (iv) (-6, -2) and (-4, -8)
- Find the distance between the centres of the circles.  
(i)  $x^2 + y^2 - 8x - 2y + 16 = 0$  and  $x^2 + y^2 - 4x - 4y - 1 = 0$   
(ii)  $x^2 + y^2 - 2x + 6y + 6 = 0$  and  $x^2 + y^2 - 5x + 6y + 15 = 0$
- State the conditions for two circles to touch each other externally and internally.
- Find the constants g, f, and c of the circles  $x^2 + y^2 - 2x + 3y - 7 = 0$  and  $x^2 + y^2 + 4x - 6y + 2 = 0$ .
- Find the equation of the circle concentric with the circle  $x^2 + y^2 + 3 = 0$  and passing through (1, 1).
- Show that the circles  $x^2 + y^2 + x + y + 1 = 0$  and  $x^2 + y^2 + x + y - 5 = 0$  are concentric.
- Show that  $2x^2 - 8xy + 8y^2 + 3x + 5y + 1 = 0$  represents a parabola.

11. Show that  $2x^2 - 16xy + 8y^2 - y + 3 = 0$  represents a hyperbola.
12. Show that  $x^2 + 2xy + 3y^2 + x - y + 1 = 0$  represents an ellipse.
13. Define conic and write the focus-directrix property.

### PART – B

1. Find the equation of the circle passing through  $(1, -4)$  and having its centre at  $(6, -2)$ .
2. Show that the line  $x + 2y + 5 = 0$  passes through the centre of the circle  $x^2 + y^2 - 6x + 8y = 0$ .
3. Find the equation of the circle concentric with the circle  $x^2 + y^2 - 2x + 5y + 1 = 0$  and passing through the point  $(2, -1)$ .
4. Find the equation of the circle concentric with the circle  $x^2 + y^2 + 8x - 4y - 23 = 0$  and having radius 3 units.
5. Find the equation of the ellipse whose focus is  $(-1, 1)$ , eccentricity is  $1/2$  and whose directrix is  $x - y + 3 = 0$ .
6. Find the equation of the parabola with
  - (i) focus  $(1, -1)$  and directrix  $x - y = 0$
  - (ii) focus  $(0, 0)$  and directrix  $x - 2y + 2 = 0$
  - (iii) focus  $(3, -4)$  and directrix  $x - y + 5 = 0$

### PART – C

1. Find the equation of the circle on the line joining the points  $(2, 3)$ ,  $(-4, 5)$  as diameter. Also find the centre and radius of the circle.
2. Show that  $(7, -5)$  lies on the circle  $x^2 + y^2 - 6x + 4y - 12 = 0$ . Find the other end of the diameter through it.
3.  $3x - y + 5 = 0$  and  $4x + 7y + 15 = 0$  are the equations of two diameters of a circle radius 4. Write down the equation of the circle.
4. Find the equation of the circle passing through the point  $(2, 4)$  and has its centre at the intersection of  $x - y = 4$  and  $2x + 3y = 7$ .
5. Find the equation of the circle two of whose diameters are  $x + 2y + 1 = 0$ ,  $y = x + 7$  and passing through the point  $(-2, 5)$ .
6. Show that the following circles touch each other:
  - (i)  $x^2 + y^2 - 25 = 0$  and  $x^2 + y^2 - 18x + 24y + 125 = 0$
  - (ii)  $x^2 + y^2 + 2x - 4y - 3 = 0$  and  $x^2 + y^2 - 8x + 6y + 7 = 0$
  - (iii)  $x^2 + y^2 - 2x + 6y + 6 = 0$  and  $x^2 + y^2 - 5x + 6y + 15 = 0$
7. Show that the following circles cut orthogonally.
  - (i)  $x^2 + y^2 - 8x - 2y + 16 = 0$  and  $x^2 + y^2 - 4x - 4y - 1 = 0$
  - (ii)  $x^2 + y^2 + 4x + 2y - 5 = 0$  and  $x^2 + y^2 + 6x - 10y + 7 = 0$
8. Show that the following conic equation represents a pair of straight line.
  - (i)  $9x^2 - 6xy + y^2 + 18x - 6y + 8 = 0$
  - (ii)  $9x^2 + 24xy + 16y^2 + 21x + 28y + 6 = 0$
  - (iii)  $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$
9. If the equation  $2x^2 + 3xy - 2y^2 - 5x + 5y + c = 0$  represents a pair of straight line. Find the value of  $c$ .

10. Find the equation of the ellipse whose

(i) focus (1, 2), directrix is  $2x - 3y + 6 = 0$ ,  $e = \frac{2}{3}$

(ii) focus (0, 0), directrix  $3x + 4y - 1 = 0$ ,  $e = \frac{5}{6}$ .

(iii) focus (1, -2) directrix  $3x - 2y + 1 = 0$ ,  $e = \frac{1}{\sqrt{2}}$

11. Find the equation of hyperbola with (i) focus (2, 2),  $e = \frac{3}{2}$  and directrix  $3x - 4y = 1$ .

(ii) focus (0, 0)  $e = \frac{5}{4}$  and directrix  $x \cos \alpha + y \sin \alpha = p$ .

(iii) focus (2, 0)  $e = 2$  and directrix  $x = y = 0$ .

### ANSWERS

#### PART - A

1. (i)  $x^2 + y^2 - 6x - 4y - 3 = 0$

(ii)  $x^2 + y^2 + 10x - 14y + 65 = 0$

(iii)  $x^2 + y^2 + 10x + 8y + 16 = 0$

(iv)  $x^2 + y^2 - 12x + 4y - 60 = 0$

2.  $(-5, 2)$ ,  $\sqrt{7}$

3. (i)  $(6, 4)$ ,  $\sqrt{50}$  (ii)  $\left(\frac{-7}{2}, \frac{-5}{2}\right)$ ,  $\frac{\sqrt{78}}{2}$  (iii)  $\left(\frac{3}{2}, -3\right)$ ,  $\frac{\sqrt{53}}{2}$  (iv)  $(0, 0)$ , 10

4. (i)  $x^2 + y^2 - 5x - 12y + 41 = 0$

(ii)  $x^2 + y^2 + x + 3y = 0$

(iii)  $x^2 + y^2 - 4x - 4y = 0$

(iv)  $x^2 + y^2 + 10x + 10y + 40 = 0$

5. (i)  $\sqrt{5}$  (ii)  $\frac{3}{2}$

7.  $g_1 = -1$ ,  $f_1 = \frac{3}{2}$ ,  $c_1 = -7$ ,  $g_2 = 2$ ,  $f_2 = -3$ ,  $c_2 = 2$  8.  $x^2 + y^2 - z = 0$

#### PART - B

1.  $x^2 + y^2 - 12x + 4y + 11 = 0$

2.  $x^2 + y^2 - 2x + 5y + 4 = 0$

4.  $x^2 + y^2 + 8x - 4y + 11 = 0$

5.  $7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$

6. (i)  $x^2 + 2xy + y^2 - 4x + 4y + 4 = 0$  (ii)  $4x^2 + 4xy + y^2 - 4x + 8y - 4 = 0$  (iii)  $x^2 + 2xy + y^2 - 22x + 26y + 25 = 0$

#### PART - C

1.  $x^2 + y^2 + 2x - 8y + 7 = 0$ ,  $(-1, 4)$ ,  $\sqrt{10}$

2.  $(-1, 1)$

3.  $x^2 + y^2 + 4x + 2y - 11 = 0$

4.  $5x^2 + 5y^2 - 38x - 2y - 32 = 0$

5.  $x^2 + y^2 + 10x - 4y + 11 = 0$

6. (i) Externally (ii) Externally (iii) internally

9.  $c = -3$

10. (i)  $101x^2 + 81y^2 + 48xy - 330x - 324y + 441 = 0$

(ii)  $27x^2 + 20y^2 - 24xy + 6x + 8y - 1 = 0$

(iii)  $17x^2 + 22y^2 + 12xy - 58x + 108y + 129 = 0$

11. (i)  $9x^2 + 216xy - 44y^2 - 346x - 472y + 791 = 0$

(ii)  $16(x^2 + y^2) - 25(x \cos \alpha + y \sin \alpha - P)^2 = 0$

(iii)  $x^2 + y^2 - 4xy + 4x - 4 = 0$

## UNIT – II

# VECTOR ALGEBRA-I

### 2.1 Introduction:

Definition of vectors – types, addition and subtraction of vectors, Properties of addition and subtraction, Position Vector, Resolution of vector in two and three dimensions, Direction cosines, direction ratios – Simple problems.

### 2.2 Scalar Product of Vectors:

Definition of scalar product of two vectors – Properties – Angle between two vectors – Simple Problems.

### 2.3 Application of Scalar Product:

Geometrical meaning of scalar product. Workdone by Force – Simple Problems.

## 2.1 INTRODUCTION

A scalar quantity or briefly a scalar, has magnitude, but is not related to any direction in space. Examples of such are mass, volume, density, temperature, work, Real numbers.

A vector quantity, or briefly a vector, has magnitude and is related to a definite direction in space. Examples of such are Displacement, velocity, acceleration, momentum, force.

A vector is a directed line segment. The length of the segment is called magnitude of the vector. The direction is indicated by an arrow joining the initial and final points of the line segment. The vector AB, i.e, joining the initial point A and the final point B in the direction of AB is denoted as  $\vec{AB}$ . The magnitude of the vector  $\vec{AB}$  is  $AB = |\vec{AB}|$ .

### Zero vector or Null vector:

A zero vector is one whose magnitude is zero, but no definite direction associated with it. For example if A is a point,  $\vec{AA}$  is a zero vector.

### Unit Vector:

A vector of magnitude one unit is called an unit vector. If  $\vec{a}$  is an unit vector, it is also denoted as  $\hat{a}$ . i.e,  $|\hat{a}| = |\vec{a}| = 1$ .

### Negative Vector:

If  $\vec{AB}$  is a vector, then the negative vector of  $\vec{AB}$  is  $\vec{BA}$ . If the direction of a vector is changed, we can get the negative vector.

$$\text{i.e, } \vec{BA} = - \vec{AB}$$

## Equal vectors:

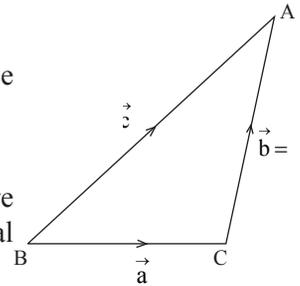
Two vectors are said to be equal, if they have the same magnitude and the same direction, but it is not required to have the same segment for the two vectors.

For example, in a parallelogram ABCD,  $\vec{AB} = \vec{CD}$  and  $\vec{AD} = \vec{BC}$ .

## Addition of two vectors:

If  $\vec{BC} = \vec{a}$ ,  $\vec{CA} = \vec{b}$  and  $\vec{BA} = \vec{c}$ , then  $\vec{BC} + \vec{CA} = \vec{BA}$  i.e.,  $\vec{a} + \vec{b} = \vec{c}$  [see figure].

If the end point of first vector and the initial point of the second vector are same, the addition of two vectors can be formed as the vector joining the initial point of the first vector and the end point of the second vector.



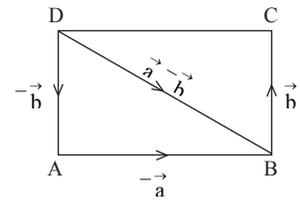
## Properties of vector addition:

- 1) Vector addition is commutative i.e.,  $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- 2) Vector addition is associative i.e.,  $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ .

## Subtraction of two vectors:

If  $\vec{AB} = \vec{a}$  and  $\vec{BC} = \vec{b}$ ,

$$\begin{aligned} \vec{a} - \vec{b} &= \vec{a} + (-\vec{b}) \\ &= \vec{AB} + \vec{CB} \\ &= \vec{AB} + \vec{DA} \quad [ \because \vec{CB} \text{ and } \vec{DA} \text{ are equal} ] \\ &= \vec{DA} + \vec{AB} \quad [ \because \text{addition is commutative} ] \\ &= \vec{DB} \end{aligned}$$



## Multiplication by a scalar:

If  $\vec{a}$  is a given vector and  $\lambda$  is a scalar, then  $\lambda \vec{a}$  is a vector whose magnitude is  $|\lambda| |\vec{a}|$  and whose direction is the same to that of  $\vec{a}$ , provided  $\lambda$  is a positive quantity. If  $\lambda$  is negative,  $\lambda \vec{a}$  is a vector whose magnitude is  $|\lambda| |\vec{a}|$  and whose direction is opposite to that of  $\vec{a}$ .

### Properties:

- 1)  $(m + n) \vec{a} = m \vec{a} + n \vec{a}$
- 2)  $m (n \vec{a}) = n (m \vec{a}) = mn \vec{a}$
- 3)  $m (\vec{a} + \vec{b}) = m \vec{a} + m \vec{b}$

## Collinear Vectors:

If  $\vec{a}$  and  $\vec{b}$  are such that they have the same or opposite directions, they are said to be collinear vectors and one is a numerical multiple of the other, i.e.,  $\vec{b} = k \vec{a}$  or  $\vec{a} = k \vec{b}$ .

## Resolution of Vectors:

Let  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  be coplanar vectors such that no two vectors are parallel. Then there exists scalars  $\alpha$  and  $\beta$  such that

$$\vec{c} = \alpha \vec{a} + \beta \vec{b}$$

Similarly, we can get constants (scalars) such that  $\vec{a} = \alpha' \vec{b} + \beta' \vec{c}$  and  $\vec{b} = \alpha'' \vec{c} + \beta'' \vec{a}$ .

If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$ ,  $\vec{d}$  are four vectors, no three of which are coplanar, then there exist scalars  $l$ ,  $m$ ,  $n$  such that

$$\vec{d} = l \vec{a} + m \vec{b} + n \vec{c},$$

### Position Vector:

If  $P$  is any point in the space and  $O$  is the origin, then  $\vec{OP}$  is called the position vector of the point  $P$ .

Let  $P$  be a point in a plane. Let  $O$  be the origin and  $\vec{i}$  and  $\vec{j}$  the unit vectors along the  $x$  and  $y$  axes in that plane. Then if  $P$  is  $(\alpha, \beta)$ , the position vector of the point  $P$  is  $\vec{OP} = \alpha \vec{i} + \beta \vec{j}$ .

Similarly if  $P$  is any point  $(x, y, z)$  in the space,  $\vec{i}$ ,  $\vec{j}$ ,  $\vec{k}$  be the unit vectors along the  $x$ ,  $y$ ,  $z$  axes in the space, the position vector of the point  $P$  is  $\vec{OP} = x \vec{i} + y \vec{j} + z \vec{k}$ . The magnitude of  $\vec{OP}$  is  $OP = |\vec{OP}| = \sqrt{x^2 + y^2 + z^2}$ .

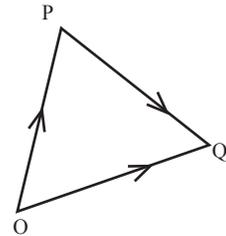
### Distance between two points:

If  $P$  and  $Q$  are two points in the space with co-ordinates  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$ , then the position vectors are  $\vec{OP} = x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k}$

$$\text{and } \vec{OQ} = x_2 \vec{i} + y_2 \vec{j} + z_2 \vec{k}$$

Now, Distance between the points  $P$  &  $Q$  is

$$\begin{aligned} PQ &= |\vec{PQ}| = |\vec{OQ} - \vec{OP}| \\ &= |(x_2 - x_1) \vec{i} + (y_2 - y_1) \vec{j} + (z_2 - z_1) \vec{k}| \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \end{aligned}$$



### Direction Cosines and Direction Ratios:

Let  $AB$  be a straight line making angles  $\alpha$ ,  $\beta$ ,  $\gamma$  with the co-ordinates axes  $x'ox$ ,  $y'oy$ ,  $z'oz$  respectively. Then  $\cos \alpha$ ,  $\cos \beta$ ,  $\cos \gamma$  are called the direction cosines of the line  $AB$  and denoted by  $l$ ,  $m$ ,  $n$ . Let  $OP$  be parallel to  $AB$  and  $P$  be  $(x, y, z)$ . Then  $OP$  also makes angles  $\alpha$ ,  $\beta$ ,  $\gamma$  with  $x$ ,  $y$  and  $z$  axes. Now,  $OP = r = \sqrt{x^2 + y^2 + z^2}$ .

$$\text{Then, } \cos \alpha = \frac{x}{r}, \cos \beta = \frac{y}{r}, \cos \gamma = \frac{z}{r}.$$

Now, sum of squares of the direction cosines of any straight line is

$$\begin{aligned} l^2 + m^2 + n^2 &= \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2 \\ &= \frac{x^2 + y^2 + z^2}{r^2} = \frac{r^2}{r^2} = 1 \end{aligned}$$

**Note:**

Let  $\hat{n}$  be the unit vector along OP.

$$\begin{aligned}\text{Then, } \hat{n} &= \frac{\vec{OP}}{|\vec{OP}|} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r} \\ &= \frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k} \\ &= l\vec{i} + m\vec{j} + n\vec{k}\end{aligned}$$

Any three numbers p, q, r proportional to the direction cosines of the straight line AB are called the direction ratios of the straight line AB.

## WORKED EXAMPLES

### PART – A

1. If position vectors of the points A and B are  $2\vec{i} + \vec{j} - \vec{k}$  and  $5\vec{i} + 4\vec{j} + 3\vec{k}$ , find  $|\vec{AB}|$ .

**Solution:**

Position vector of the point A,

$$\text{i.e., } \vec{OA} = 2\vec{i} + \vec{j} - \vec{k}$$

Position vector of the point B,

$$\text{i.e., } \vec{OB} = 5\vec{i} + 4\vec{j} - 3\vec{k}$$

$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$= (5\vec{i} + 4\vec{j} - 3\vec{k}) - (2\vec{i} + \vec{j} - \vec{k})$$

$$= 3\vec{i} + 3\vec{j} - 2\vec{k}$$

$$\begin{aligned}\therefore AB &= |\vec{AB}| = \sqrt{3^2 + 3^2 + (-2)^2} \\ &= \sqrt{9 + 9 + 4} = \sqrt{22} \text{ units}\end{aligned}$$

2. Find the unit vector along  $4\vec{i} - 5\vec{j} + 7\vec{k}$ .

**Solution:**

$$\text{Let } \vec{a} = 4\vec{i} - 5\vec{j} + 7\vec{k}$$

$$|\vec{a}| = \sqrt{4^2 + (-5)^2 + 7^2}$$

$$= \sqrt{16 + 25 + 49}$$

$$= \sqrt{90}$$

$$\therefore \text{Unit vector along } \vec{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{4\vec{i} - 5\vec{j} + 7\vec{k}}{\sqrt{90}}$$

3. Find the direction cosines of the vector  $2\vec{i} + 3\vec{j} - 4\vec{k}$ .

**Solution:**

$$\text{Let } \vec{a} = 2\vec{i} + 3\vec{j} - 4\vec{k}$$

$$\begin{aligned} r = |\vec{a}| &= \sqrt{2^2 + 3^2 + (-4)^2} \\ &= \sqrt{4 + 9 + 16} \\ &= \sqrt{29} \end{aligned}$$

Direction cosines of  $\vec{a}$  are

$$\cos \alpha = \frac{x}{r} = \frac{2}{\sqrt{29}}$$

$$\cos \beta = \frac{y}{r} = \frac{3}{\sqrt{29}}$$

$$\cos \gamma = \frac{z}{r} = \frac{-4}{\sqrt{29}}$$

4. Find the direction ratios of the vector  $\vec{i} + 2\vec{j} - \vec{k}$ .

**Solution:**

The direction ratios of  $\vec{i} + 2\vec{j} - \vec{k}$  are 1, 2, -1.

### PART - B

1. If the vectors  $\vec{a} = 2\vec{i} - 3\vec{j}$  and  $\vec{b} = -6\vec{i} + m\vec{j}$  are collinear, find the value of them.

**Solution:**

$\vec{a} = 2\vec{i} - 3\vec{j}$  and  $\vec{b} = -6\vec{i} + m\vec{j}$ . By given,  $\vec{a}$  and  $\vec{b}$  are collinear.

$$\therefore \vec{a} = t\vec{b}$$

$$\begin{aligned} \vec{a} &= 2\vec{i} - 3\vec{j} \\ &= t(-6\vec{i} + m\vec{j}) \end{aligned}$$

$$2\vec{i} - 3\vec{j} = -6t\vec{i} + mt\vec{j}$$

Comparing coefficient of  $\vec{i}$

$$2 = -6t \Rightarrow t = \frac{-1}{3}$$

Comparing coefficient of  $\vec{j}$

$$-3 = mt$$

$$-3 = m\left(\frac{-1}{3}\right)$$

$$\therefore m = 9$$

2. If A (2, 3, -4) and B (1, 0, 5) are two points, find the direction cosines of  $\vec{AB}$ .

**Solution:**

By given, the points are

$$A(2, 3, -4) \text{ and } B(1, 0, 5)$$

$$\therefore \text{Position vectors are } \vec{OA} = 2\vec{i} + 3\vec{j} - 4\vec{k}, \quad \vec{OB} = \vec{i} + 5\vec{k},$$

$$\therefore \vec{AB} = \vec{OB} - \vec{OA}$$

$$= (\vec{i} + 5\vec{k}) - (2\vec{i} + 3\vec{j} - 4\vec{k})$$

$$= -\vec{i} - 3\vec{j} + 9\vec{k}$$

$$r = |\vec{AB}| = \sqrt{(-1)^2 + (-3)^2 + 9^2}$$

$$= \sqrt{1 + 9 + 81} = \sqrt{91}$$

$\therefore$  Direction cosines of  $\vec{AB}$  are

$$\cos \alpha = \frac{-1}{\sqrt{91}} \quad \cos \beta = \frac{-3}{\sqrt{91}} \quad \cos \gamma = \frac{9}{\sqrt{91}}$$

### PART - C

1. Show that the points whose position vectors  $2\vec{i} + 3\vec{j} - 5\vec{k}$ ,  $3\vec{i} + \vec{j} - 2\vec{k}$  and  $6\vec{i} - 5\vec{j} + 7\vec{k}$  are collinear.

**Solution:**

$$\vec{OA} = 2\vec{i} + 3\vec{j} - 5\vec{k}$$

$$\vec{OB} = 3\vec{i} + \vec{j} - 2\vec{k}$$

$$\vec{OC} = 6\vec{i} - 5\vec{j} + 7\vec{k}$$

$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$= (3\vec{i} + \vec{j} - 2\vec{k}) - (2\vec{i} + 3\vec{j} - 5\vec{k})$$

$$= \vec{i} - 2\vec{j} + 3\vec{k} \quad \dots\dots\dots (1)$$

$$\vec{BC} = \vec{OC} - \vec{OB}$$

$$= (6\vec{i} - 5\vec{j} + 7\vec{k}) - (3\vec{i} + \vec{j} - 2\vec{k})$$

$$= 3\vec{i} - 6\vec{j} + 9\vec{k}$$

$$= 3(\vec{i} - 2\vec{j} + 3\vec{k})$$

$$= 3\vec{AB} \quad \text{[From (1)]}$$

i.e.,  $\vec{BC} = 3\vec{AB}$

$\therefore \vec{AB}$  and  $\vec{BC}$  are parallel vectors and B is the common point of these two vectors.

$\therefore$  The given points A, B and C are collinear.

2. Prove that the points A (2, 4, -1), B (4, 5, 1) and C (3, 6, -3) form the vertices of a right angled isosceles triangle.

**Solution:**

$$\vec{OA} = 2\vec{i} + 4\vec{j} - \vec{k} \quad \vec{OB} = 4\vec{i} + 5\vec{j} + \vec{k} \quad \vec{OC} = 3\vec{i} + 6\vec{j} - 3\vec{k}$$

$$\begin{aligned} \vec{AB} &= \vec{OB} - \vec{OA} \\ &= (4\vec{i} + 5\vec{j} + \vec{k}) - (2\vec{i} + 4\vec{j} - \vec{k}) \\ &= 2\vec{i} + \vec{j} + 2\vec{k} \end{aligned}$$

$$\begin{aligned} \vec{BC} &= \vec{OC} - \vec{OB} \\ &= (3\vec{i} + 6\vec{j} - 3\vec{k}) - (4\vec{i} + 5\vec{j} + \vec{k}) \\ &= -\vec{i} + \vec{j} - 4\vec{k} \end{aligned}$$

$$\begin{aligned} \vec{AC} &= \vec{OC} - \vec{OA} \\ &= (3\vec{i} + 6\vec{j} - 3\vec{k}) - (2\vec{i} + 4\vec{j} - \vec{k}) \\ &= \vec{i} + 2\vec{j} - 2\vec{k} \end{aligned}$$

$$\text{Now, } AB = |\vec{AB}| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9}$$

$$BC = |\vec{BC}| = \sqrt{(-1)^2 + 1^2 + (-4)^2} = \sqrt{1 + 1 + 16} = \sqrt{18}$$

$$AC = |\vec{AC}| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{1 + 4 + 4} = \sqrt{9}$$

$$AB = AC = \sqrt{9} = 3$$

$$\& AB^2 + AC^2 = 9 + 9 = 18 = BC^2$$

$\therefore$  Triangle ABC is an isosceles triangle as well as a right angled triangle with  $\hat{A} = 90^\circ$ .

3. Prove that the position vectors  $4\vec{i} + 5\vec{j} + 6\vec{k}$ ,  $5\vec{i} + 6\vec{j} + 4\vec{k}$  and  $6\vec{i} + 4\vec{j} + 5\vec{k}$  form the vertices of an equilateral triangle.

**Solution:**

$$\text{Let } \vec{OA} = 4\vec{i} + 5\vec{j} + 6\vec{k} \quad \vec{OB} = 5\vec{i} + 6\vec{j} + 4\vec{k} \quad \vec{OC} = 6\vec{i} + 4\vec{j} + 5\vec{k}$$

$$\begin{aligned} \vec{AB} &= \vec{OB} - \vec{OA} = (5\vec{i} + 6\vec{j} + 4\vec{k}) - (4\vec{i} + 5\vec{j} + 6\vec{k}) \\ &= \vec{i} + \vec{j} - 2\vec{k} \end{aligned}$$

$$\begin{aligned} \vec{BC} &= \vec{OC} - \vec{OB} = (6\vec{i} + 4\vec{j} + 5\vec{k}) - (5\vec{i} + 6\vec{j} + 4\vec{k}) \\ &= \vec{i} - 2\vec{j} + \vec{k} \end{aligned}$$

$$\begin{aligned} \vec{AC} &= \vec{OC} - \vec{OA} = (6\vec{i} + 4\vec{j} + 5\vec{k}) - (4\vec{i} + 5\vec{j} + 6\vec{k}) \\ &= 2\vec{i} - \vec{j} - \vec{k} \end{aligned}$$

$$\text{Now, } AB = |\vec{AB}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{1 + 1 + 4} = \sqrt{6}$$

$$BC = |\vec{BC}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{1 + 4 + 1} = \sqrt{6}$$

$$AC = |\vec{AC}| = \sqrt{2^2 + (-1)^2 + (-1)^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$$

$$\text{Here, } AB = BC = CA = \sqrt{6}$$

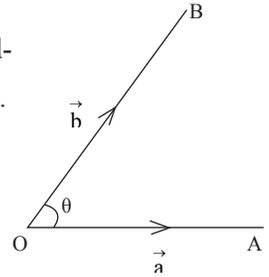
$\therefore$  The given triangle is an equilateral triangle, [since the sides are equal].

## 2.2 SCALAR PRODUCT OF VECTORS OR DOT PRODUCT

If the product of two vectors  $\vec{a}$  and  $\vec{b}$  gives a scalar, it is called scalar product of the vectors  $\vec{a}$  and  $\vec{b}$  and is denoted as  $\vec{a} \cdot \vec{b}$  (pronounce as  $\vec{a}$  dot  $\vec{b}$ ).

If the angle between two vectors  $\vec{a}$  and  $\vec{b}$  is  $\theta$ , then

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$



### Properties of scalar product:

1. If  $\theta$  is an acute angle  $\vec{a} \cdot \vec{b}$  is positive and if  $\theta$  is an obtuse angle  $\vec{a} \cdot \vec{b}$  is negative.

2. Scalar product is commutative.

$$\text{i.e., } \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta = |\vec{b}| |\vec{a}| \cos \theta = \vec{b} \cdot \vec{a}$$

3. If  $\vec{a}$  and  $\vec{b}$  are (non zero) perpendicular vectors, then the angle  $\theta$  between them is  $90^\circ$ .

$$\therefore \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos 90^\circ = 0 \quad [\because \cos 90^\circ = 0]$$

If  $\vec{a} \cdot \vec{b} = 0$ , either  $\vec{a} = 0$  or  $\vec{b} = 0$  or  $\vec{a}$  and  $\vec{b}$  are perpendicular vector.

$\therefore$  The condition for two perpendicular vectors  $\vec{a}$  and  $\vec{b}$  is  $\vec{a} \cdot \vec{b} = 0$ .

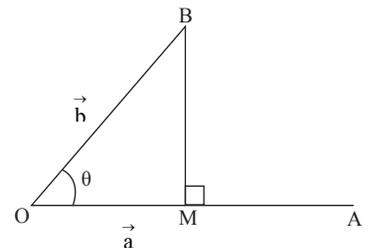
4. If  $\vec{a}$  and  $\vec{b}$  are parallel vectors,  $\theta = 0$  or  $180^\circ$ .

$$\begin{aligned} \therefore \vec{a} \cdot \vec{b} &= |\vec{a}| |\vec{b}| \cos 0 \\ &= |\vec{a}| |\vec{b}| \quad [\because \cos 0 = 1] \end{aligned}$$

As a special case,  $\therefore \vec{a} \cdot \vec{a} = |\vec{a}| |\vec{a}| = |\vec{a}|^2 = a^2$

5. Let  $\vec{OA} = \vec{a}$ ,  $\vec{OB} = \vec{b}$ . Draw  $BM$  perpendicular to  $OA$ . ( $BM$  perpendicular to  $OA$ )

Let  $\theta$  be the angle between  $\vec{a}$  and  $\vec{b}$ .  
i.e.,  $\angle BOA = \theta$ .



Now,  $OM$  is the projection of  $\vec{b}$  on  $\vec{a}$ .

From the right angled triangle  $BOM$ ,

$$\cos \theta = \frac{OM}{OB} = \frac{OM}{|\vec{b}|}$$

$$\therefore OM = |\vec{b}| \cos \theta$$

$$= \frac{|\vec{a}| |\vec{b}| \cos \theta}{|\vec{a}|} \quad [\text{Multiplying Nr and Dr by } |\vec{a}|]$$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \quad [\text{By definition of scalar product}]$$

$$\therefore \text{The projection of } \vec{b} \text{ on } \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

Similarly, the projection of  $\vec{a}$  on  $\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

6.  $\vec{i}, \vec{j}, \vec{k}$  are the unit vectors along the x, y and z axes respectively.

$$\therefore \vec{i} \cdot \vec{i}, \vec{j} \cdot \vec{j}, \vec{k} \cdot \vec{k} = 1 \quad [\text{using property 4}]$$

$$\left. \begin{aligned} \text{Also } \vec{i} \cdot \vec{j} &= \vec{j} \cdot \vec{i} = 0 \\ \vec{j} \cdot \vec{k} &= \vec{k} \cdot \vec{j} = 0 \\ \vec{k} \cdot \vec{i} &= \vec{i} \cdot \vec{k} = 0 \end{aligned} \right\} \quad [\text{using property 3}]$$

Hence,

•	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	1	0	0
$\vec{j}$	0	1	0
$\vec{k}$	0	0	1

7. If  $\vec{a}, \vec{b}$  and  $\vec{c}$  are three vectors,

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

8. If  $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$  &  $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$

$$\vec{a} \cdot \vec{b} = (a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}) \cdot (b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k})$$

$$= a_1 b_1 \vec{i} \cdot \vec{i} + a_1 b_2 \vec{i} \cdot \vec{j} + a_1 b_3 \vec{i} \cdot \vec{k} + a_2 b_1 \vec{j} \cdot \vec{i} +$$

$$a_2 b_2 \vec{j} \cdot \vec{j} + a_2 b_3 \vec{j} \cdot \vec{k} + a_3 b_1 \vec{k} \cdot \vec{i} + a_3 b_2 \vec{k} \cdot \vec{j} + a_3 b_3 \vec{k} \cdot \vec{k}$$

$$= a_1 b_1 + 0 + 0 + 0 + a_2 b_2 + 0 + 0 + 0 + a_3 b_3 \quad [\text{By property 6}]$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

9. Angle between two vectors

$$\text{We know } \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

$$\therefore \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

$$\text{If } \vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k} \text{ \& } \vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

$$\text{then } \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}$$

$$10. (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = a^2 + b^2 + 2\vec{a} \cdot \vec{b}$$

$$11. (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = a^2 + b^2 - 2\vec{a} \cdot \vec{b}$$

$$12. (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = a^2 - b^2$$

## WORKED EXAMPLES

### PART – A

1. Find the scalar product of the two vectors  $3\vec{i} + 4\vec{j} + 5\vec{k}$  and  $2\vec{i} + 3\vec{j} + \vec{k}$ .

**Solution:**

$$\text{Let } \vec{a} = 3\vec{i} + 4\vec{j} + 5\vec{k}$$

$$\vec{b} = 2\vec{i} + 3\vec{j} + \vec{k}$$

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (3\vec{i} + 4\vec{j} + 5\vec{k}) \cdot (2\vec{i} + 3\vec{j} + \vec{k}) \\ &= 3(2) + 4(3) + 5(1) \\ &= 6 + 12 + 5 \\ &= 23 \end{aligned}$$

2. Prove that the vectors  $3\vec{i} - \vec{j} + 5\vec{k}$  and  $-6\vec{i} + 2\vec{j} + 4\vec{k}$  are perpendicular.

**Solution:**

$$\text{Let } \vec{a} = 3\vec{i} - \vec{j} + 5\vec{k}$$

$$\vec{b} = -6\vec{i} + 2\vec{j} + 4\vec{k}$$

$$\begin{aligned} \text{Now, } \vec{a} \cdot \vec{b} &= (3\vec{i} - \vec{j} + 5\vec{k}) \cdot (-6\vec{i} + 2\vec{j} + 4\vec{k}) \\ &= 3(-6) + (-1)2 + 5(4) \\ &= -18 - 2 + 20 \\ &= 0 \end{aligned}$$

$\therefore$  The vectors  $\vec{a}$  and  $\vec{b}$  are perpendicular vectors.

3. Find the value of 'p' if the vectors  $2\vec{i} + p\vec{j} - \vec{k}$  and  $3\vec{i} + 4\vec{j} + 2\vec{k}$  are perpendicular.

**Solution:**

$$\text{Let } \vec{a} = 2\vec{i} + p\vec{j} - \vec{k}$$

$$\vec{b} = 3\vec{i} + 4\vec{j} + 2\vec{k}$$

$\vec{a}$  and  $\vec{b}$  are perpendicular

$$\therefore \vec{a} \cdot \vec{b} = 0$$

$$\text{i.e., } (2\vec{i} + p\vec{j} - \vec{k}) \cdot (3\vec{i} + 4\vec{j} + 2\vec{k}) = 0$$

$$\text{i.e., } 2(3) + p(4) + (-1)2 = 0$$

$$\text{i.e., } 6 + 4p - 2 = 0$$

$$\text{i.e., } 4p = 2 - 6 = -4$$

$$\therefore p = -\frac{4}{4} = -1$$

**PART – B**

1. Find the projection of  $2\vec{i} + \vec{j} + 2\vec{k}$  on  $\vec{i} + 2\vec{j} + 2\vec{k}$ .

**Solution:**

$$\begin{aligned}\text{Let } \vec{a} &= 2\vec{i} + \vec{j} + 2\vec{k} \\ \vec{b} &= \vec{i} + 2\vec{j} + 2\vec{k}\end{aligned}$$

$$\begin{aligned}\text{Projection of } \vec{a} \text{ on } \vec{b} &= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} \\ &= \frac{(2\vec{i} + \vec{j} + 2\vec{k}) \cdot (\vec{i} + 2\vec{j} + 2\vec{k})}{|\vec{i} + 2\vec{j} + 2\vec{k}|} \\ &= \frac{2(1) + 1(2) + 2(2)}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{2 + 2 + 4}{\sqrt{1 + 4 + 4}} \\ &= \frac{8}{\sqrt{9}} = \frac{8}{3}\end{aligned}$$

**PART – C**

1. Find the angle between the two vectors  $\vec{i} + \vec{j} + \vec{k}$  and  $3\vec{i} - \vec{j} + 2\vec{k}$ .

**Solution:**

$$\begin{aligned}\text{Let } \vec{a} &= \vec{i} + \vec{j} + \vec{k} \\ \vec{b} &= 3\vec{i} - \vec{j} + 2\vec{k}\end{aligned}$$

$$\begin{aligned}\vec{a} \cdot \vec{b} &= (\vec{i} + \vec{j} + \vec{k}) \cdot (3\vec{i} - \vec{j} + 2\vec{k}) \\ &= 1(3) + 1(-1) + 1(2) \\ &= 3 - 1 + 2 \\ &= 4\end{aligned}$$

$$|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$|\vec{b}| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{9 + 1 + 4} = \sqrt{14}$$

Let  $\theta$  be the angle between  $\vec{a}$  &  $\vec{b}$

$$\begin{aligned}\therefore \cos\theta &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \\ &= \frac{4}{\sqrt{3} \cdot \sqrt{14}} = \frac{4}{\sqrt{42}}\end{aligned}$$

$$\therefore \theta = \cos^{-1}\left(\frac{4}{\sqrt{42}}\right)$$

2. Show that the vectors  $-3\vec{i} + 2\vec{j} - \vec{k}$ ,  $\vec{i} - 3\vec{j} + 5\vec{k}$  and  $2\vec{i} + \vec{j} - 4\vec{k}$  form a right angled triangle, using scalar product.

**Solution:**

Let the sides of the triangle be

$$\vec{a} = -3\vec{i} + 2\vec{j} - \vec{k}$$

$$\vec{b} = \vec{i} - 3\vec{j} + 5\vec{k}$$

$$\vec{c} = 2\vec{i} + \vec{j} - 4\vec{k}$$

$$\begin{aligned}\text{Now, } \vec{a} \cdot \vec{b} &= (-3\vec{i} + 2\vec{j} - \vec{k}) \cdot (\vec{i} - 3\vec{j} + 5\vec{k}) \\ &= -3(1) + (+2)(-3) + (-1)(5) \\ &= -3 - 6 - 5 = -14\end{aligned}$$

$$\begin{aligned}\vec{b} \cdot \vec{c} &= (\vec{i} - 3\vec{j} + 5\vec{k}) \cdot (2\vec{i} + \vec{j} - 4\vec{k}) \\ &= 1(2) + (-3)(1) + 5(-4) \\ &= 2 - 3 - 20 = -21\end{aligned}$$

$$\begin{aligned}\vec{c} \cdot \vec{a} &= (2\vec{i} + \vec{j} - 4\vec{k}) \cdot (-3\vec{i} + 2\vec{j} - \vec{k}) \\ &= 2(-3) + 1(+2) + (-4)(-1) \\ &= -6 + 2 + 4 = 0\end{aligned}$$

Now,  $\vec{c} \cdot \vec{a} = 0$  and

$$\begin{aligned}\vec{a} + \vec{c} &= (-3\vec{i} + 2\vec{j} - \vec{k}) + (2\vec{i} + \vec{j} - 4\vec{k}) \\ &= -\vec{i} + 3\vec{j} - 5\vec{k} \\ &= -\vec{b}\end{aligned}$$

$\therefore$  The sides  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  form a triangle and the angle between  $\vec{a}$  and  $\vec{c}$  is  $90^\circ$  hence right angled triangle.

3. Prove that the vectors  $2\vec{i} - 2\vec{j} + \vec{k}$ ,  $\vec{i} + 2\vec{j} + 2\vec{k}$ ,  $2\vec{i} + \vec{j} - 2\vec{k}$  are perpendicular to each other. (one another).

**Solution:**

$$\vec{a} = 2\vec{i} - 2\vec{j} + \vec{k}$$

$$\vec{b} = \vec{i} + 2\vec{j} + 2\vec{k}$$

$$\vec{c} = 2\vec{i} + \vec{j} - 2\vec{k}$$

$$\begin{aligned}\text{Now, } \vec{a} \cdot \vec{b} &= (2\vec{i} - 2\vec{j} + \vec{k}) \cdot (\vec{i} + 2\vec{j} + 2\vec{k}) \\ &= 2(1) + (-2)(2) + 1(2) = 2 - 4 + 2 = 0\end{aligned}$$

$\therefore \vec{a}$  is perpendicular to  $\vec{b}$

$$\begin{aligned}\vec{b} \cdot \vec{c} &= (\vec{i} + 2\vec{j} + 2\vec{k}) \cdot (2\vec{i} + \vec{j} - 2\vec{k}) \\ &= 1(2) + 2(1) + 2(-2) = 2 + 2 - 4 = 0\end{aligned}$$

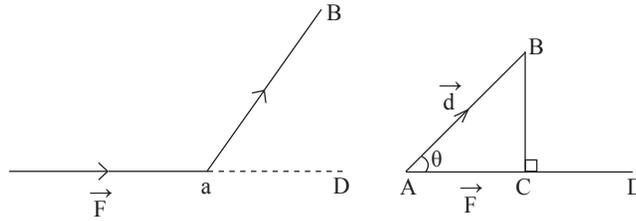
$\therefore \vec{b}$  is perpendicular to  $\vec{c}$

$$\begin{aligned}\vec{c} \cdot \vec{a} &= (2\vec{i} + \vec{j} - 2\vec{k}) \cdot (2\vec{i} - 2\vec{j} + \vec{k}) \\ &= 2(2) + 1(-2) + (-2)(1) = 4 - 2 - 2 = 0\end{aligned}$$

$\therefore \vec{c}$  is perpendicular to  $\vec{a}$

$\therefore$  The three vectors are perpendicular to one another.

## 2.3 APPLICATION OF SCALAR PRODUCT



A Force  $\vec{F}$  acting on a particle, displaces that particle from the point A to the point B. Hence the vector  $\vec{AB}$  is called the displacement vector  $\vec{d}$  of the particle due to the force  $\vec{F}$ .

The force  $\vec{F}$  acting on the particle does work when the particle is displaced in the direction which is not perpendicular to the force  $\vec{F}$ . The workdone is a scalar quantity proportional to the force and the resolved part of the displacement in the direction of the force. We choose the unit quantity of the work as the work done when a particle, acted on by unit force, is displaced unit distance in the direction of the force.

Hence, if  $\vec{F}$ ,  $\vec{d}$  are the vectors representing the force and the displacement respectively, inclined at an angle  $\theta$ , the measure of workdone is

$$|\vec{F}| |\vec{d}| \cos\theta = \vec{F} \cdot \vec{d}$$

$$\text{i.e., workdone, } W = \vec{F} \cdot \vec{d}$$

### WORKED EXAMPLES

#### PART – A

- $3\vec{i} + 5\vec{j} + 7\vec{k}$  is the force acting on a particle giving the displacement  $2\vec{i} - \vec{j} + \vec{k}$ . Find the Workdone.

**Solution:**

$$\text{The force } \vec{F} = 3\vec{i} + 5\vec{j} + 7\vec{k}$$

$$\text{Displacement } \vec{d} = 2\vec{i} - \vec{j} + \vec{k}$$

$$\begin{aligned} \therefore \text{Workdone, } W &= \vec{F} \cdot \vec{d} = (3\vec{i} + 5\vec{j} + 7\vec{k}) \cdot (2\vec{i} - \vec{j} + \vec{k}) \\ &= 3(2) + 5(-1) + 7(1) = 6 - 5 + 7 = 8 \end{aligned}$$

#### PART – B

- A particle moves from the point  $(1, -2, 5)$  to the point  $(3, 4, 6)$  due to the force  $4\vec{i} + \vec{j} - 3\vec{k}$  acting on it. Find the workdone.

**Solution:**

$$\text{The force } \vec{F} = 4\vec{i} + \vec{j} - 3\vec{k}$$

The particle moves from A  $(1, -2, 5)$  to B  $(3, 4, 6)$ .

$$\begin{aligned} \therefore \text{Displacement vector, } \vec{d} &= \vec{AB} \\ &= \vec{OB} - \vec{OA} \\ &= (3\vec{i} + 4\vec{j} + 6\vec{k}) - (\vec{i} - 2\vec{j} + 5\vec{k}) \\ &= 2\vec{i} + 6\vec{j} + \vec{k} \end{aligned}$$

$$\begin{aligned} \therefore \text{Workdone, } W &= \vec{F} \cdot \vec{d} \\ &= (4\vec{i} + \vec{j} - 3\vec{k}) \cdot (2\vec{i} + 6\vec{j} + \vec{k}) \\ &= 4(2) + 1(6) + (-3)1 = 8 + 6 - 3 = 11 \end{aligned}$$

**PART – C**

1. If a particle moves from  $3\vec{i} - \vec{j} + \vec{k}$  to  $2\vec{i} - 3\vec{j} + \vec{k}$  due to the forces  $2\vec{i} + 5\vec{j} - 3\vec{k}$  and  $4\vec{i} + 3\vec{j} + 2\vec{k}$ . Find the workdone of the forces.

**Solution:**

$$\text{The forces are } \vec{F}_1 = 2\vec{i} + 5\vec{j} - 3\vec{k}$$

$$\vec{F}_2 = 4\vec{i} + 3\vec{j} + 2\vec{k}$$

$$\therefore \text{Total force, } \vec{F} = \vec{F}_1 + \vec{F}_2$$

$$= (2\vec{i} + 5\vec{j} - 3\vec{k}) + (4\vec{i} + 3\vec{j} + 2\vec{k}) = 6\vec{i} + 8\vec{j} - \vec{k}$$

$$\text{The particle moves from } \vec{OA} = 3\vec{i} - \vec{j} + \vec{k} \text{ to } \vec{OB} = 2\vec{i} - 3\vec{j} + \vec{k}$$

$\therefore$  The displacement vector,

$$\vec{d} = \vec{AB} = \vec{OB} - \vec{OA} = (2\vec{i} - 3\vec{j} + \vec{k}) - (3\vec{i} - \vec{j} + \vec{k}) = -\vec{i} - 2\vec{j}$$

$$\therefore \text{Workdone, } W = \vec{F} \cdot \vec{d}$$

$$= (6\vec{i} + 8\vec{j} - \vec{k}) \cdot (-\vec{i} - 2\vec{j})$$

$$6(-1) + 8(-2) - 1(0) = -6 - 16 + 0 = -22$$

**EXERCISE****PART – A**

- If A and B are two points whose position vectors are  $\vec{i} - 2\vec{j} + 2\vec{k}$  and  $3\vec{i} + 5\vec{j} - 7\vec{k}$  respectively find  $|\vec{AB}|$ .
- If  $\vec{OA} = \vec{i} + 2\vec{j} + 2\vec{k}$  and  $\vec{OB} = 2\vec{i} - 3\vec{j} + \vec{k}$ , find  $|\vec{AB}|$ .
- A and B are (3, 2, -1) and (7, 5, 2). Find  $|\vec{AB}|$ .
- Find the unit vector along  $2\vec{i} - \vec{j} + 4\vec{k}$ .
- Find the unit vector along  $\vec{i} + 2\vec{j} - 3\vec{k}$ .
- Find the direction cosines of the vector  $2\vec{i} - 3\vec{j} + 4\vec{k}$ .
- Find the modulus and direction cosines of the vector  $4\vec{i} - 3\vec{j} + \vec{k}$ .
- Find the direction cosines and direction ratios of the vector  $\vec{i} - 2\vec{j} + 3\vec{k}$ .
- Find the scalar product of the vectors.
  - $3\vec{i} + 4\vec{j} - 5\vec{k}$  and  $2\vec{i} + \vec{j} + \vec{k}$
  - $\vec{i} - \vec{j} + \vec{k}$  and  $-2\vec{i} + 3\vec{j} - 5\vec{k}$
  - $\vec{i} + \vec{j}$  and  $\vec{k} + \vec{i}$
  - $\vec{i} + 2\vec{j} - 3\vec{k}$  and  $\vec{i} - 2\vec{j} + \vec{k}$
- Prove that the two vectors are perpendicular to each other.
  - $3\vec{i} - \vec{j} + 5\vec{k}$  and  $-\vec{i} + 2\vec{j} + \vec{k}$
  - $8\vec{i} + 7\vec{j} - \vec{k}$  and  $3\vec{i} - 3\vec{j} + 3\vec{k}$
  - $\vec{i} - 3\vec{j} + 5\vec{k}$  and  $-2\vec{i} + 6\vec{j} + 4\vec{k}$
  - $2\vec{i} + 3\vec{j} + \vec{k}$  and  $4\vec{i} - 2\vec{j} - 2\vec{k}$

11. If the two vectors are perpendicular, find the value of  $p$ .

(i)  $p\vec{i} + 3\vec{j} + 4\vec{k}$  and  $2\vec{i} + 2\vec{j} - 5\vec{k}$

(ii)  $p\vec{i} + 2\vec{j} + 3\vec{k}$  and  $-\vec{i} + 3\vec{j} - 4\vec{k}$

(iii)  $2\vec{i} + p\vec{j} - \vec{k}$  and  $3\vec{i} - 4\vec{j} + \vec{k}$

(iv)  $\vec{i} + 2\vec{j} - \vec{k}$  and  $p\vec{i} + \vec{j}$

(v)  $\vec{i} - 2\vec{j} - 4\vec{k}$  and  $2\vec{i} - p\vec{j} + 3\vec{k}$

12. Define the scalar product of two vectors  $\vec{a}$  and  $\vec{b}$ .

13. Write down the condition for two vectors to be perpendicular.

14. Write down the formula for the projection of  $\vec{a}$  on  $\vec{b}$ .

15. If a force  $\vec{F}$  acts on a particle giving the displacement  $\vec{d}$ , write down the formula for the workdone by the force.

### PART – B

1. The position vectors of A and B are  $\vec{i} + 3\vec{j} - 4\vec{k}$  and  $2\vec{i} + \vec{j} - 5\vec{k}$  find unit vector along  $\overline{AB}$ .

2. If  $\overline{OA} = 2\vec{i} + 3\vec{j} - 4\vec{k}$  and  $\overline{OB} = \vec{i} + \vec{j} - 2\vec{k}$  find the direction cosines of the vector  $\overline{AB}$ .

3. If A is (2, 3, -1) and B is (4, 0, 7) find the direction ratios of  $\overline{AB}$ .

4. If the vectors  $\vec{i} + 2\vec{j} + \vec{k}$  and  $-2\vec{i} + k\vec{j} - 2\vec{k}$  are collinear, find the value of  $k$ .

5. Find the projection of

(i)  $2\vec{i} + \vec{j} - 2\vec{k}$  on  $\vec{i} - 2\vec{j} - 2\vec{k}$

(ii)  $3\vec{i} + 4\vec{j} + 12\vec{k}$  on  $\vec{i} + 2\vec{j} + 2\vec{k}$

(iii)  $\vec{j} + \vec{k}$  on  $\vec{i} + \vec{j}$

(iv)  $8\vec{i} + 3\vec{j} + 2\vec{k}$  on  $\vec{i} + \vec{j} + \vec{k}$

### PART – C

1. Prove that the triangle having the following position vectors of the vertices form an equilateral triangle:

(i)  $4\vec{i} + 2\vec{j} + 3\vec{k}$ ,  $2\vec{i} + 3\vec{j} + 4\vec{k}$ ,  $3\vec{i} + 4\vec{j} + 2\vec{k}$

(ii)  $3\vec{i} + \vec{j} + 2\vec{k}$ ,  $\vec{i} + 2\vec{j} + 3\vec{k}$ ,  $2\vec{i} + 3\vec{j} + \vec{k}$

(iii)  $2\vec{i} + 3\vec{j} + 4\vec{k}$ ,  $5\vec{i} + 2\vec{j} + 3\vec{k}$ ,  $3\vec{i} + 5\vec{j} + 2\vec{k}$

2. Prove that the triangles whose vertices have following position vectors form an isosceles triangles.

(i)  $3\vec{i} - \vec{j} - 2\vec{k}$ ,  $5\vec{i} + \vec{j} - 3\vec{k}$ ,  $6\vec{i} - \vec{j} - \vec{k}$

(ii)  $-7\vec{i} - 10\vec{k}$ ,  $4\vec{i} - 9\vec{j} - 6\vec{k}$ ,  $\vec{i} - 6\vec{j} - 6\vec{k}$

(iii)  $7\vec{i} + 10\vec{k}$ ,  $3\vec{i} - 4\vec{j} + 6\vec{k}$ ,  $9\vec{i} - 4\vec{j} + 6\vec{k}$

3. Prove that the following position vectors of the vertices of a triangle form a right angled triangle.
- $3\vec{i} + \vec{j} - 5\vec{k}$ ,  $4\vec{i} + 3\vec{j} - 7\vec{k}$ ,  $5\vec{i} + 2\vec{j} - 3\vec{k}$
  - $2\vec{i} - \vec{j} + \vec{k}$ ,  $3\vec{i} - 4\vec{j} - 4\vec{k}$ ,  $\vec{i} - 3\vec{j} - 5\vec{k}$
  - $2\vec{i} + 4\vec{j} + 3\vec{k}$ ,  $4\vec{i} + \vec{j} - 4\vec{k}$ ,  $6\vec{i} + 5\vec{j} - \vec{k}$
4. Prove that the following vectors are collinear.
- $2\vec{i} + \vec{j} - \vec{k}$ ,  $4\vec{i} + 3\vec{j} - 5\vec{k}$ ,  $\vec{i} + \vec{k}$
  - $\vec{i} + 2\vec{j} + 4\vec{k}$ ,  $4\vec{i} + 8\vec{j} + \vec{k}$ ,  $3\vec{i} + 6\vec{j} + 2\vec{k}$
  - $2\vec{i} - \vec{j} + 3\vec{k}$ ,  $3\vec{i} - 5\vec{j} + \vec{k}$ ,  $-\vec{i} + 11\vec{j} + 9\vec{k}$
5. Find the angle between the following the vectors.
- $2\vec{i} - 3\vec{j} + 2\vec{k}$  and  $-\vec{i} + \vec{j} - \vec{k}$
  - $4\vec{i} + 3\vec{j} + \vec{k}$  and  $2\vec{i} - \vec{j} + 2\vec{k}$
  - $3\vec{i} + \vec{j} - \vec{k}$  and  $\vec{i} - \vec{j} - 2\vec{k}$
6. If the position vectors of A, B and C are  $\vec{i} + 2\vec{j} - \vec{k}$ ,  $2\vec{i} + 3\vec{k}$ ,  $3\vec{i} - \vec{j} + 2\vec{k}$ , find the angle between the vectors  $\vec{AB}$  and  $\vec{BC}$ .
7. Show that the following position vectors of the points form a right angled triangle.
- $3\vec{i} - 2\vec{j} + \vec{k}$ ,  $\vec{i} - 3\vec{j} + 4\vec{k}$ ,  $2\vec{i} + \vec{j} - 4\vec{k}$
  - $2\vec{i} + 4\vec{j} - \vec{k}$ ,  $4\vec{i} + 5\vec{j} - \vec{k}$ ,  $3\vec{i} + 6\vec{j} - 3\vec{k}$
  - $3\vec{i} - 2\vec{j} + \vec{k}$ ,  $\vec{i} - 3\vec{j} + 5\vec{k}$ ,  $2\vec{i} + \vec{j} - 4\vec{k}$
8. Due to the force  $2\vec{i} - 3\vec{j} + \vec{k}$ , a particle is displaced from the point  $\vec{i} + 2\vec{j} + 3\vec{k}$  to  $-2\vec{i} + 4\vec{j} + \vec{k}$ . Find the work done.
9. A particle is displaced from A (3, 0, 2) to B (-6, -1, 3) due to the force  $\vec{F} = 15\vec{i} + 10\vec{j} + 15\vec{k}$ . Find the work done.
10.  $\vec{F} = 2\vec{i} - 3\vec{j} + 4\vec{k}$  displaces a particle from origin to (1, 2, -1). Find the workdone of the force.
11. Two forces  $4\vec{i} + \vec{j} - 3\vec{k}$  and  $3\vec{i} + \vec{j} - \vec{k}$  displaces a particle from the point (1, 2, 3) to (5, 4, 1). Find the work done.
12. A particle is moved from  $5\vec{i} - 5\vec{j} - 7\vec{k}$  to  $6\vec{i} + 2\vec{j} - 2\vec{k}$  due to the three forces  $10\vec{i} - \vec{j} + 11\vec{k}$ ,  $4\vec{i} + 5\vec{j} + 6\vec{k}$  and  $-2\vec{i} + \vec{j} - 9\vec{k}$ . Find the workdone.
13. When a particle is moved from the point (1, 1, 1) to (2, 1, 3) by a force  $\vec{i} + \vec{j} + \vec{k}$ , the work done is 4. Find the value of A.
14. A force  $2\vec{i} + \vec{j} + \lambda\vec{k}$  displaces a particle from the point (1, 1, 1) to (2, 2, 2) giving the workdone 5. Find the value of  $\lambda$ .
15. Find the value of p, if a force  $2\vec{i} - 3\vec{j} + 4\vec{k}$  displaces a particle from (1, p, 3) to (2, 0, 5) giving the work done 17.

**ANSWERS****PART - A**

$$1) 2\bar{i} + 7\bar{j} - 9\bar{k} \quad 2) \sqrt{42} \quad 3) \sqrt{34} \quad 4) \frac{2\bar{i} - \bar{j} + 4\bar{k}}{\sqrt{21}} \quad 5) \frac{\bar{i} + 2\bar{j} - 3\bar{k}}{\sqrt{14}}$$

$$6) \frac{2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{4}{\sqrt{29}} \quad 7) \sqrt{26}, \frac{4}{\sqrt{26}}, \frac{-3}{\sqrt{26}}, \frac{1}{\sqrt{26}} \quad 8) \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, 1, -2, 3$$

$$9) \text{i) } 5 \quad \text{ii) } -10 \quad \text{iii) } 1 \quad \text{iv) } -6 \quad 11) \text{i) } p = 7 \quad \text{ii) } p = -6 \quad \text{iii) } p = \frac{5}{4} \quad \text{iv) } p = -2, \quad \text{v) } p = 5$$

$$12) |\bar{a}| |\bar{b}| \cos \theta \quad 13) \bar{a} \cdot \bar{b} = 0 \quad 14) \frac{\bar{a} \cdot \bar{b}}{|\bar{b}|} \quad 15) \bar{f} \cdot \bar{d}$$

**PART - B**

$$1) \frac{\bar{i} - 2\bar{j} - \bar{k}}{\sqrt{6}} \quad 2) -\frac{1}{3}, \frac{-2}{3}, \frac{2}{3} \quad 3) 2, -3, 8 \quad 4) K = -4,$$

$$5) \text{i) } \frac{4}{3} \quad \text{ii) } \frac{35}{3} \quad \text{iii) } \frac{1}{\sqrt{2}} \quad \text{iv) } \frac{13}{\sqrt{3}}$$

**PART - C**

$$5) \text{i) } \text{Cos}^{-1}\left(\frac{-7}{\sqrt{51}}\right) \quad \text{ii) } \text{Cos}^{-1}\left(\frac{-7}{\sqrt{234}}\right) \quad \text{iii) } \text{Cos}^{-1}\left(\frac{4}{\sqrt{16}}\right) \quad 6) \text{Cos}^{-1}\left(\frac{20}{\sqrt{462}}\right)$$

$$9) 14 \quad 10) 130 \quad 11) 8 \quad 12) 40 \quad 13) 37 \quad 14) \lambda = 2 \quad 15) \lambda = 2 \quad 16) p = \frac{7}{3}$$

# UNIT – III

## VECTOR ALGEBRA - III

### 3.1 VECTOR PRODUCT OF TWO VECTORS

Definition of vector product of two vectors. Geometrical meaning. Properties – Angle between two vectors – unit vector perpendicular to two vectors. Simple problems.

### 3.2 APPLICATION OF VECTOR PRODUCT OF TWO VECTORS & SCALAR TRIPLE PRODUCT

Definition of moment of a force. Definition of scalar product of three vectors – Geometrical meaning – Coplanar vectors. Simple problems.

### 3.3 VECTOR TRIPLE PRODUCT & PRODUCT OF MORE VECTORS

Definition of Vector Triple product, Scalar and Vector product of four vectors Simple Problems.

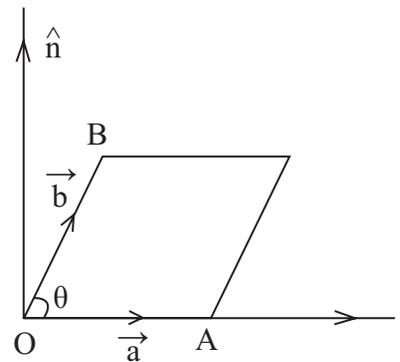
## 3.1 VECTOR PRODUCT OF TWO VECTORS

### Definition:

Let  $\vec{a}$  and  $\vec{b}$  be two vectors and ‘ $\theta$ ’ be the angle between them. The vector product of  $\vec{a}$  and  $\vec{b}$  is denoted as  $\vec{a} \times \vec{b}$  and it is defined as a vector  $|\vec{a}| |\vec{b}| \sin \theta \hat{n}$  where  $\hat{n}$  is the unit vector perpendicular to both  $\vec{a}$  and  $\vec{b}$  such that  $\vec{a}, \vec{b}, \hat{n}$  are in right handed system. Thus

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$$

The vector product is called as cross product.



### Geometrical meaning of vector product

Let  $\vec{OA} = \vec{a}$ ,  $\vec{OB} = \vec{b}$  and ‘ $\theta$ ’ be the angle between  $\vec{a}$  and  $\vec{b}$ . Complete the parallelogram OACB with  $\vec{OA}$  and  $\vec{OB}$  as its adjacent sides.

Draw  $BN \perp OA$ .

From the right angled  $\Delta ONB$ .

$$\frac{BN}{OB} = \sin \theta \Rightarrow BN = OB \cdot \sin \theta$$

$$\Rightarrow BN = |\vec{b}| \sin \theta$$

By definition,  $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$$

$$= OA \times BN$$

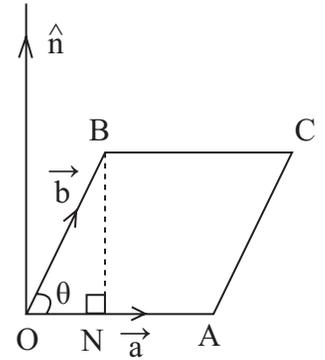
$$= \text{base} \times \text{height}$$

$$= \text{Area of parallelogram OACB.}$$

$\therefore |\vec{a} \times \vec{b}| = \text{Area of parallelogram with } |\vec{a}| \text{ and } |\vec{b}| \text{ as adjacent sides.}$

Also, area of  $\Delta OAB = \frac{1}{2}$  (area of parallelogram OACB)

$$= \frac{1}{2} |\vec{OA} \times \vec{OB}| = \frac{1}{2} |\vec{a} \times \vec{b}|$$



**Results :**

1. The area of a parallelogram with adjacent sides  $\vec{a}$  and  $\vec{b}$  is  $A = |\vec{a} \times \vec{b}|$ .
2. The vector area of parallelogram with adjacent sides  $\vec{a}$  and  $\vec{b}$  is  $|\vec{a} \times \vec{b}|$
3. The area of a triangle with adjacent sides  $\vec{a}$  and  $\vec{b}$  is  $A = \frac{1}{2} |\vec{a} \times \vec{b}|$
4. The area of triangle ABC is given by either  $\frac{1}{2} |\vec{AB} \times \vec{AC}|$  or  $\frac{1}{2} |\vec{BC} \times \vec{BA}|$  or  $\frac{1}{2} |\vec{CA} \times \vec{CB}|$ .

**Properties of vector product**

**1. Vector product is not commutative.**

If  $\vec{a}$  and  $\vec{b}$  are any two vectors then  $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$  however  $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

**2. Vector product of collinear or parallel vectors:**

If  $\vec{a}$  and  $\vec{b}$  are collinear or parallel then  $\theta = 0, \pi$

For  $\theta = 0, \pi$  then  $\sin \theta = 0$

$$\therefore \vec{a} \times \vec{b} = |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta \hat{n} = \vec{0}$$

$$\Rightarrow \vec{a} \times \vec{b} = \vec{0}$$

Thus,  $\vec{a}$  and  $\vec{b}$  are collinear or parallel  $\Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$

Note : If  $\vec{a} \times \vec{b} = \vec{0}$  then

- (i)  $\vec{a}$  is a zero vector and  $\vec{b}$  is any vector.
- (ii)  $\vec{b}$  is a zero vector and  $\vec{a}$  is any vector.
- (iii)  $\vec{a}$  and  $\vec{b}$  are parallel (collinear)

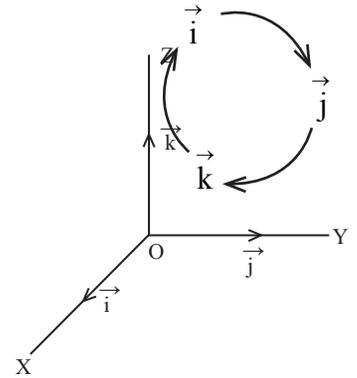
3. Cross product of equal vectors

$$\begin{aligned} \vec{a} \times \vec{a} &= |\vec{a}| |\vec{a}| \sin 0 \hat{n} \\ &= |\vec{a}| |\vec{a}| (0) \hat{n} \\ &= \vec{0} \end{aligned}$$

∴  $\vec{a} \times \vec{a} = \vec{0}$  for every non zero vector  $\vec{a}$ .

4) Cross product of unit vectors  $\vec{i}, \vec{j}, \vec{k}$

Let  $\vec{i}, \vec{j}, \vec{k}$  be the three mutually perpendicular unit vectors. The involvement of these three unit vectors in vector product as



follows : By property(3),  $\vec{i} \times \vec{i} = \vec{0}, \vec{j} \times \vec{j} = \vec{0}, \vec{k} \times \vec{k} = \vec{0}$   
 also  $\vec{i} \times \vec{j} = |\vec{i}| |\vec{j}| \sin 90^\circ \vec{k} = 1.1.1. \vec{k} = \vec{k}$  similarly,  $\vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j}$  and  
 $\vec{j} \times \vec{i} = -\vec{k}, \vec{k} \times \vec{j} = -\vec{i}, \vec{i} \times \vec{k} = -\vec{j}$  etc. This can be shown as follows.

$\vec{i}$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{j}$	$\vec{0}$	$\vec{k}$	$-\vec{i}$
$\vec{k}$	$-\vec{k}$	$\vec{0}$	$\vec{i}$
$\vec{i}$	$\vec{i}$	$-\vec{i}$	$\vec{0}$

5. If m is any scalar and  $\vec{a}, \vec{b}$  are two vectors inclined at an angle 'θ' then  $m \vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times m \vec{b}$

6. Distributing of vector product nor vector addition :

Let  $\vec{a}, \vec{b}, \vec{c}$  be any three vectors then

(i)  $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$  [Left distributivity]

(ii)  $(\vec{b} + \vec{c}) \times \vec{a} = (\vec{b} \times \vec{a}) + (\vec{c} \times \vec{a})$  [Right distributivity]

7. Vector product in determinant from

Let  $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$  and  $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$  then  $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$\Rightarrow \vec{a} \times \vec{b} = \vec{i} (a_2 b_3 - b_2 a_3) - \vec{j} (a_1 b_3 - b_1 a_3) + \vec{k} (a_1 b_2 - b_1 a_2)$

## 8. Angle between two vectors

Let  $\vec{a}$ ,  $\vec{b}$  be two vectors inclined at an angle ' $\theta$ ' then,

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$$

$$\Rightarrow \sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|}$$

$$\Rightarrow \theta = \sin^{-1} \left[ \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|} \right]$$

## 9. Unit vector perpendicular to two vectors

Let  $\vec{a}$ ,  $\vec{b}$  be two non zero, non parallel vectors and ' $\theta$ ' be the angle between them

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n} \text{ ————— (1)}$$

$$\text{also, } |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta \text{ ————— (2)}$$

$$\therefore (1) \div (2) \Rightarrow \frac{|\vec{a}| |\vec{b}| \sin \theta \hat{n}}{|\vec{a}| |\vec{b}| \sin \theta} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

$$\Rightarrow \hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

Note that,  $-\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$  is also a unit vector

perpendicular to  $\vec{a}$  and  $\vec{b}$ .

$$\therefore \text{Unit vector perpendicular to } \vec{a} \text{ and } \vec{b} \text{ are } \pm \hat{n} = \pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

## WORKED EXAMPLES

## PART - A

1. Prove that  $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = 2(\vec{b} \times \vec{a})$

**Solution :**

$$\begin{aligned} \text{L.H.S. : } & (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) \\ &= \vec{a} \times \vec{a} \times -\vec{a} \times \vec{b} + \vec{b} \times \vec{a} - \vec{b} \times \vec{b} \\ &= \vec{0} + \vec{b} \times \vec{a} + \vec{b} \times \vec{a} - \vec{0} \\ &= 2(\vec{b} \times \vec{a}) = \text{R.H.S.} \end{aligned}$$

2. Prove that  $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}$ .

**Solution :** L.H.S:  $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$   
 $= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} + \vec{c} \times \vec{b}$   
 $= \vec{a} \times \vec{b} - \vec{c} \times \vec{a} + \vec{b} \times \vec{c} - \vec{a} \times \vec{b} + \vec{c} \times \vec{a} - \vec{b} \times \vec{c}$   
 $= \vec{0}$  R.H.S.

3. If  $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$ ,  $\vec{b} = \vec{i} + 2\vec{j} + 3\vec{k}$  find  $\vec{a} \times \vec{b}$ .

**Solution :**

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} \\ &= \vec{i} \begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \\ &= \vec{i} (-3-2) - \vec{j} (6-1) + \vec{k} (4+1) \\ &= \vec{i} (-5) - \vec{j} (5) + \vec{k} (5) \\ \Rightarrow \vec{a} \times \vec{b} &= -5\vec{i} - 5\vec{j} + 5\vec{k} \end{aligned}$$

4. If  $|\vec{a}| = 2$ ,  $|\vec{b}| = 7$  and  $|\vec{a} \times \vec{b}| = 7$  find the angle between  $\vec{a}$  and  $\vec{b}$ .

**Solution :**

$$\begin{aligned} \text{We have, } \sin \theta &= \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|} = \frac{7}{2 \times 7} \\ \Rightarrow \sin \theta &= \frac{1}{2} \Rightarrow \theta = \sin^{-1} \left( \frac{1}{2} \right) = 30^\circ \end{aligned}$$

### PART - B

1. Find the area of the parallelogram whose adjacent sides are  $\vec{i} + \vec{j} + \vec{k}$  and  $3\vec{i} - \vec{k}$ .

**Solution :**

$$\text{Let } \vec{a} = \vec{i} + \vec{j} + \vec{k} \text{ \& } \vec{b} = 3\vec{i} - \vec{k}$$

**Formula :**

The area of parallelogram from whose adjacent sides are

$$\vec{a} \text{ \& } \vec{b} \text{ is } A = |\vec{a} \times \vec{b}| \text{ square units. } \text{————— (1)}$$

$$\begin{aligned} \text{Now, } \vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 3 & 0 & -1 \end{vmatrix} \\ &= \vec{i} \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} \end{aligned}$$

$$= \vec{i}(-1-0) - \vec{j}(-1-3) + \vec{k}(0-3)$$

$$\Rightarrow \vec{a} \times \vec{b} = -\vec{i} + 4\vec{j} - 3\vec{k}$$

also,  $|\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + (4)^2 + (-3)^2} = \sqrt{1+16+9}$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{26}$$

area of parallelogram is  $|\vec{a} \times \vec{b}| = \sqrt{26}$  square units.

2. Find the area of the triangle whose adjacent sides are  $2\vec{i} - \vec{j} + \vec{k}$  and  $3\vec{i} + 4\vec{j} - \vec{k}$

**Solution :**

Let  $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$  &  $\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$

Formula : The area of the triangle whose adjacent sides are  $\vec{a}$  &  $\vec{b}$  is

$$A = \frac{1}{2} |\vec{a} \times \vec{b}| \quad \text{square units} \quad \text{———— (1)}$$

Now,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} -1 & 1 \\ 4 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$$

$$= \vec{i}(1-4) - \vec{j}(-2-3) + \vec{k}(8+3)$$

$$= \vec{i}(-3) - \vec{j}(-5) + \vec{k}(11)$$

$$\Rightarrow \vec{a} \times \vec{b} = -3\vec{i} + 5\vec{j} + 11\vec{k}$$

also,  $|\vec{a} \times \vec{b}| = \sqrt{(-3)^2 + (5)^2 + (11)^2} = \sqrt{9+25+121}$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{155}$$

$\therefore$  (1) becomes, area of triangle is  $A = \frac{1}{2} \sqrt{155}$  square units.

3. Prove that  $(\vec{a} \times \vec{b})^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$

**Solution :**

$$\text{L.H.S. } (\vec{a} \times \vec{b})^2 + (\vec{a} \cdot \vec{b})^2$$

$$= \left[ |\vec{a}| |\vec{b}| \sin \theta \hat{n} \right]^2 + \left[ |\vec{a}| |\vec{b}| \cos \theta \right]^2$$

$$\begin{aligned}
&= |\vec{a}|^2 |\vec{b}|^2 \sin^2 \theta (\hat{n})^2 + |\vec{a}|^2 |\vec{b}|^2 \cos^2 \theta \\
&= |\vec{a}|^2 |\vec{b}|^2 [\sin^2 \theta + \cos^2 \theta] \quad [ \because (\hat{n})^2 = 1 ] \\
&= |\vec{a}|^2 |\vec{b}|^2 [1] \\
&= |\vec{a}|^2 |\vec{b}|^2 = \text{R.H.S.}
\end{aligned}$$

4. If  $|\vec{a}| = 13$ ,  $|\vec{b}| = 5$  and  $\vec{a} \cdot \vec{b} = 60$  then find  $|\vec{a} \times \vec{b}|$ .

**Solution :**

We have,

$$\begin{aligned}
|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 &= |\vec{a}|^2 |\vec{b}|^2 \\
\Rightarrow |\vec{a} \times \vec{b}|^2 + (60)^2 &= (13)^2 \cdot (5)^2 \\
\Rightarrow |\vec{a} \times \vec{b}|^2 + 3600 &= 169 \times 25 \\
\Rightarrow |\vec{a} \times \vec{b}|^2 &= 4225 - 3600 \\
\Rightarrow |\vec{a} \times \vec{b}|^2 &= 625 \\
\therefore |\vec{a} \times \vec{b}| &= \sqrt{625} = 25.
\end{aligned}$$

### PART - C

1. Find the unit vector perpendicular to each of the vectors  $\vec{i} + 2\vec{j} + 3\vec{k}$  and  $\vec{i} - \vec{j} - \vec{k}$ . Also find the sine of the angle between these two vectors.

**Solution :**

$$\text{Given : } \vec{a} = \vec{i} + 2\vec{j} + 3\vec{k} \quad \& \quad \vec{b} = \vec{i} - \vec{j} - \vec{k}$$

$$\text{Formula : (i) Unit vector : } \hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

$$\text{(ii) Sine of the angle : } \sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|}$$

Now,

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{vmatrix} \\
&= \vec{i} \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} \\
&= \vec{i} (-2+3) - \vec{j} (-1-3) + \vec{k} (-1-2) \\
&= \vec{i} (1) - \vec{j} (-4) + \vec{k} (-3) \\
\Rightarrow \vec{a} \times \vec{b} &= \vec{i} + 4\vec{j} - 3\vec{k}
\end{aligned}$$

$$\text{also, } |\vec{a} \times \vec{b}| = \sqrt{(1)^2 + (4)^2 + (-3)^2} = \sqrt{1+16+9}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{26}$$

$$\text{Again, } |\vec{a}| = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{1+4+9} = \sqrt{14}$$

$$\& |\vec{b}| = \sqrt{(1)^2 + (-1)^2 + (-1)^2} = \sqrt{1+1+1} = \sqrt{3}$$

Hence,

$$(i) \text{ Unit vector : } \hat{n} = \frac{\vec{i} + 4\vec{j} - 3\vec{k}}{\sqrt{26}}$$

$$(ii) \sin \theta = \frac{\sqrt{26}}{\sqrt{14} \sqrt{3}} \Rightarrow \theta = \sin^{-1} \left[ \frac{\sqrt{26}}{\sqrt{14} \sqrt{3}} \right]$$

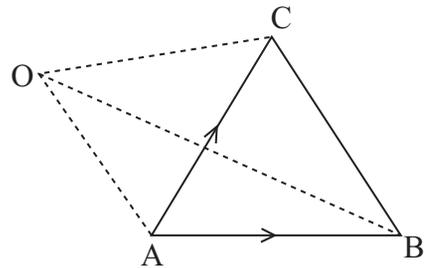
2. Find the area of the triangle formed by the points whose position vectors are  $2\vec{i} + 3\vec{j} + 4\vec{k}$ ,  $3\vec{i} + 4\vec{j} + 2\vec{k}$ ,  $4\vec{i} + 2\vec{j} + 3\vec{k}$ .

**Solution :**

$$\text{Let } \vec{OA} = 2\vec{i} + 3\vec{j} + 4\vec{k}$$

$$\vec{OB} = 3\vec{i} + 4\vec{j} + 2\vec{k}$$

$$\& \vec{OC} = 4\vec{i} + 2\vec{j} + 3\vec{k}$$



be the given position vectors of the vertices of  $\Delta ABC$ .

$$\text{Formula : Area of triangle } ABC = \frac{1}{2} |\vec{AB} \times \vec{AC}| \text{ ————— (1)}$$

Now,

$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$= (3\vec{i} + 4\vec{j} + 2\vec{k}) - (2\vec{i} + 3\vec{j} + 4\vec{k})$$

$$= 3\vec{i} + 4\vec{j} + 2\vec{k} - 2\vec{i} - 3\vec{j} - 4\vec{k}$$

$$\Rightarrow \vec{AB} = \vec{i} + \vec{j} - 2\vec{k}$$

$$\text{also, } \vec{AC} = \vec{OC} - \vec{OA}$$

$$= (4\vec{i} + 2\vec{j} + 3\vec{k}) - (2\vec{i} + 3\vec{j} + 4\vec{k})$$

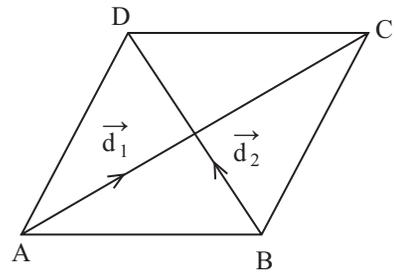
$$= 4\vec{i} + 2\vec{j} + 3\vec{k} - 2\vec{i} - 3\vec{j} - 4\vec{k}$$

$$\Rightarrow \vec{AC} = 2\vec{i} - \vec{j} - \vec{k}$$

$$\begin{aligned} \therefore \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix} \\ &= \vec{i} \begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \\ &= \vec{i} (-1-2) - \vec{j} (-1+4) + \vec{k} (-1-2) \\ &= \vec{i} (-3) - \vec{j} (+3) + \vec{k} (-3) \\ \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} &= -3\vec{i} - 3\vec{j} - 3\vec{k} \end{aligned}$$

$$\begin{aligned} \text{and } |\overrightarrow{AB} \times \overrightarrow{AC}| &= \sqrt{(-3)^2 + (-3)^2 + (-3)^2} \\ &= \sqrt{9+9+9} = \sqrt{27} \\ \Rightarrow |\overrightarrow{AB} \times \overrightarrow{AC}| &= 3\sqrt{3} \end{aligned}$$

$$\text{Area of } \Delta ABC = \frac{1}{2} (3\sqrt{3}) = \frac{3\sqrt{3}}{2} \text{ square units.}$$



3. Find the area of the parallelogram whose diagonals are represented by  $3\vec{i} + \vec{j} - 2\vec{k}$  and  $\vec{i} - 3\vec{j} + 4\vec{k}$ .

**Solution :**

$$\text{Let } \overrightarrow{d_1} = 3\vec{i} + \vec{j} - 2\vec{k} \text{ \& } \overrightarrow{d_2} = \vec{i} - 3\vec{j} + 4\vec{k}$$

be the given diagonals of parallelogram ABCD.

$$\text{Formula : Area of parallelogram} = \frac{1}{2} |\overrightarrow{d_1} \times \overrightarrow{d_2}| \text{ square units. ——— (1)}$$

Now,

$$\begin{aligned} \overrightarrow{d_1} \times \overrightarrow{d_2} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} \\ &= \vec{i} \begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix} - \vec{j} \begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} + \vec{k} \begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} \\ &= \vec{i} (4-6) - \vec{j} (12+2) + \vec{k} (-9-1) \\ &= \vec{i} (-2) - \vec{j} (14) + \vec{k} (-10) \\ \Rightarrow \overrightarrow{d_1} \times \overrightarrow{d_2} &= -2\vec{i} - 14\vec{j} - 10\vec{k} \end{aligned}$$

$$\text{also, } \left| \vec{d}_1 \times \vec{d}_2 \right| = \sqrt{(-2)^2 + (-14)^2 + (-10)^2} = \sqrt{4 + 196 + 100} = \sqrt{300}$$

$$\Rightarrow \left| \vec{d}_1 \times \vec{d}_2 \right| = 10\sqrt{3}$$

∴ (1) becomes,

$$\text{Area of parallelogram} = \frac{1}{2} [10\sqrt{3}] = 5\sqrt{3} \text{ square units}$$

4. If  $\vec{a} = \vec{i} + 3\vec{j} - 2\vec{k}$  and  $\vec{b} = -\vec{i} + 3\vec{k}$  and  $\vec{a} \times \vec{b}$ . Verify that  $\vec{a}$  is perpendicular to  $\vec{a} \times \vec{b}$  and  $\vec{b}$  is perpendicular to  $\vec{a} \times \vec{b}$ .

**Solution :**

$$\text{Given : } \vec{a} = \vec{i} + 3\vec{j} - 2\vec{k} \text{ \& } \vec{b} = -\vec{i} + 3\vec{k}$$

Now,

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix} \\ &= \vec{i} \begin{vmatrix} 3 & -2 \\ 0 & 3 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} \\ &= \vec{i} (9+0) - \vec{j} (3-2) + \vec{k} (0+3) \\ &= \vec{i} (9) - \vec{j} (1) + \vec{k} (3) \end{aligned}$$

$$\Rightarrow \vec{a} \times \vec{b} = 9\vec{i} - \vec{j} + 3\vec{k}$$

(i) To show  $\vec{a} \perp (\vec{a} \times \vec{b})$  :

$$\begin{aligned} \vec{a} \cdot (\vec{a} \times \vec{b}) &= (\vec{i} + 3\vec{j} - 2\vec{k}) \cdot (9\vec{i} - \vec{j} + 3\vec{k}) \\ &= (1 \times 9) + (3 \times -1) + (-2 \times 3) \\ &= 9 - 3 - 6 \end{aligned}$$

$$\therefore \vec{a} \cdot (\vec{a} \times \vec{b}) = 0 \Rightarrow \vec{a} \perp (\vec{a} \times \vec{b})$$

(ii) To show  $\vec{b} \perp (\vec{a} \times \vec{b})$  :

$$\begin{aligned} \vec{b} \cdot (\vec{a} \times \vec{b}) &= (-\vec{i} + 3\vec{k}) \cdot (9\vec{i} - \vec{j} + 3\vec{k}) \\ &= (-1 \times 9) + (3 \times 3) \\ &= -9 + 9 \end{aligned}$$

$$\therefore \vec{b} \cdot (\vec{a} \times \vec{b}) = 0 \Rightarrow \vec{b} \perp (\vec{a} \times \vec{b})$$

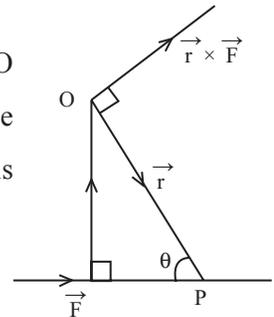
## 3.2 APPLICATION OF VECTOR PRODUCT OF TWO VECTORS AND SCALAR TRIPLE PRODUCT

**Definition :**

### Moment (or) Torque of a force about a point

Let O be any point and  $\vec{r}$  be the position vector relative to the point O of any point P on the line of action of the force  $\vec{F}$ . The moment of the force about the point O is defined as  $\vec{M} = \vec{r} \times \vec{F}$ . The magnitude of the moment is  $|\vec{M}| = |\vec{r} \times \vec{F}| = |\vec{r}| |\vec{F}| \sin \theta$ .

The moment of the force is also called as Torque of the force.



### Definition : Scalar triple Product

Let  $\vec{a}, \vec{b}, \vec{c}$  be any three vectors. The scalar product of the two vectors  $\vec{a} \times \vec{b}$  and  $\vec{c}$ , i.e.,  $(\vec{a} \times \vec{b}) \cdot \vec{c}$  is called the **scalar triple product** and it is denoted by  $[\vec{a}, \vec{b}, \vec{c}]$ . The scalar triple product is also called as box product (or) mixed product.

### Geometrical meaning of Scalar triple product :

Let  $\vec{a}, \vec{b}, \vec{c}$  be three non collinear vectors. Consider a parallelepiped having co-terminus edges OA, OB, OC so that  $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}$  &  $\vec{OC} = \vec{c}$ . Then  $\vec{a} \times \vec{b}$  is a vector perpendicular to the plane containing  $\vec{a}$  and  $\vec{b}$ .

Let  $\phi$  be the angle between  $\vec{c}$  and  $\vec{a} \times \vec{b}$ .

Draw CL perpendicular to the base OADB.

Let CL = the height of the parallelepiped. Here CL and  $\vec{a} \times \vec{b}$  are perpendicular to the same plane.

$$\Rightarrow CL \parallel \vec{a} \times \vec{b} \Rightarrow \angle OCL = \phi$$

In right angled  $\Delta OCL$ ,

$$\cos \phi = \frac{CL}{OC} = \frac{CL}{|\vec{c}|} \Rightarrow CL = |\vec{c}| \cos \phi$$

Also, base area of the parallelepiped

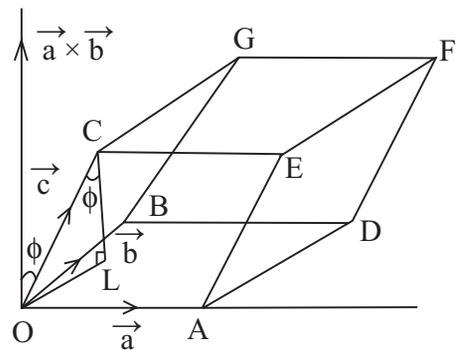
= the area of the parallelogram with  $\vec{a}$  and  $\vec{b}$  as adjacent sides =  $|\vec{a} \times \vec{b}|$ .

$\therefore$  By definition of scalar product,

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = |\vec{a} \times \vec{b}| |\vec{c}| \cos \phi$$

$$= (\text{Base area}) \times (\text{height})$$

$$= V, \text{ the volume of the parallelepiped with co-terminus edges } \vec{a}, \vec{b}, \vec{c}.$$



**Properties of Scalar triple product :**

1. Let  $\vec{a}, \vec{b}, \vec{c}$  represents co-terminous edges of a parallelopiped in right handed system then its volume  $V = (\vec{a} \times \vec{b}) \cdot \vec{c}$

Similarly  $\vec{b}, \vec{c}, \vec{a}$  and  $\vec{c}, \vec{a}, \vec{b}$  are co-terminous edges of the same parallelopiped in right handed system then

$$V = (\vec{b} \times \vec{c}) \cdot \vec{a} \text{ \& } V = (\vec{c} \times \vec{a}) \cdot \vec{b}$$

Hence,  $V = (\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$  ——— (1)

Since the scalar product is commutative then (1) becomes,

$$V = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) \text{ ————— (2)}$$

From (1) & (2),

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$(\vec{b} \times \vec{c}) \cdot \vec{a} = \vec{b} \cdot (\vec{c} \times \vec{a})$$

$$(\vec{c} \times \vec{a}) \cdot \vec{b} = \vec{c} \cdot (\vec{a} \times \vec{b})$$

In scalar triple product the dot and cross are inter changeable. Due to this property,

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}) = [\vec{a} \ \vec{b} \ \vec{c}]$$

$$\therefore [\vec{a} \ \vec{b} \ \vec{c}] = [\vec{b} \ \vec{c} \ \vec{a}] = [\vec{c} \ \vec{a} \ \vec{b}]$$

- 2) The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not the magnitude.

$$[\vec{a} \ \vec{b} \ \vec{c}] = -[\vec{b} \ \vec{a} \ \vec{c}] = [\vec{c} \ \vec{b} \ \vec{a}]$$

- 3) The scalar triple product is zero if any two of the vectors are equal.

$$\text{i.e. } [\vec{a} \ \vec{a} \ \vec{b}] = 0; [\vec{a} \ \vec{b} \ \vec{a}] = 0; [\vec{b} \ \vec{b} \ \vec{c}] = 0 \text{ etc.}$$

- 4) For any three vectors  $\vec{a}, \vec{b}, \vec{c}$  and the scalar  $\lambda$  then  $[\lambda \vec{a} \ \vec{b} \ \vec{c}] = \lambda [\vec{a} \ \vec{b} \ \vec{c}]$

- 5) The scalar product of three vectors is zero if, any two of them are parallel (or) collinear.

- 6) Coplaner Vectors : The necessary and sufficient condition for three non-zero non-collinear vectors

$$\vec{a} \ \vec{b} \ \vec{c} \text{ to be coplanar is } [\vec{a} \ \vec{b} \ \vec{c}] = 0$$

$$\text{i.e., } \vec{a} \ \vec{b} \ \vec{c} \text{ are coplanar } = [\vec{a} \ \vec{b} \ \vec{c}] = 0.$$

Note : If  $[\vec{a} \ \vec{b} \ \vec{c}] = 0$  then

- (i) Atleast one of the vectors  $\vec{a}, \vec{b}, \vec{c}$  is a zero vector.

- (ii) Any two of the vectors  $\vec{a}, \vec{b}, \vec{c}$  are parallel.

- (iii) The three vectors  $\vec{a}, \vec{b}, \vec{c}$  are coplanar.

- 7) For any three vectors  $\vec{a}, \vec{b}, \vec{c}$  then  $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ .

**Scalar triple product in terms of Components :**

Let  $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$ ,  $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$  &  $\vec{c} = c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}$  then

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = [\vec{a} \ \vec{b} \ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Proof :

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\Rightarrow \vec{b} \times \vec{c} = \vec{i} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \vec{j} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \vec{k} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}) \cdot \left[ \vec{i} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \vec{j} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \vec{k} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right]$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\Rightarrow \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = [\vec{a} \ \vec{b} \ \vec{c}]$$

**WORKED EXAMPLES****PART - A**

1. If  $\vec{F} = 2\vec{i} - 3\vec{j} + \vec{k}$  and  $\vec{r} = \vec{i} + 2\vec{j} + 4\vec{k}$  find torque.

**Solution :**

$$\text{Torque} = \vec{r} \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 4 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}$$

$$= \vec{i} (2 + 12) - \vec{j} (1 - 8) + \vec{k} (-3 - 4)$$

$$= \vec{i} (14) - \vec{j} (-7) + \vec{k} (-7)$$

$$\Rightarrow \text{Torque} = 14\vec{i} + 7\vec{j} - 7\vec{k}$$

2. Find the value of  $[\vec{i} \ \vec{j} \ \vec{k}]$

**Solution :**

$$\begin{aligned} [\vec{i} \ \vec{j} \ \vec{k}] &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \\ &= 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1(1-0) = 1 \end{aligned}$$

3. Find the value of  $[\vec{i} + \vec{j}, \vec{j} + \vec{k}, \vec{k} + \vec{i}]$

**Solution :**

$$\begin{aligned} [\vec{i} + \vec{j}, \vec{j} + \vec{k}, \vec{k} + \vec{i}] &= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \\ &= 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ &= 1(1-0) - 1(0-1) \\ &= 1+1 = 2 \end{aligned}$$

### PART - B

1. Find the scalar triple product of the vectors  $\vec{i} - 3\vec{j} + 3\vec{k}$ ,  $2\vec{i} + \vec{j} - \vec{k}$  and  $\vec{j} + \vec{k}$ .

**Solution :**

$$\text{Let } \vec{a} = \vec{i} - 3\vec{j} + 3\vec{k}, \vec{b} = 2\vec{i} + \vec{j} - \vec{k} \text{ \& } \vec{c} = \vec{j} + \vec{k}$$

$$\begin{aligned} \therefore [\vec{a} \ \vec{b} \ \vec{c}] &= \begin{vmatrix} 1 & -3 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} \\ &= 1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \\ &= 1(1+1) + 3(2+0) + 3(2-0) \\ &= 1(2) + 3(2) + 3(2) \\ &= 2+6+6 = 14 \end{aligned}$$

2. If the edges  $\vec{a} = -3\vec{i} + 7\vec{j} + 5\vec{k}$ ,  $\vec{b} = -5\vec{i} + 7\vec{j} - 3\vec{k}$  and  $\vec{c} = 7\vec{i} - 5\vec{j} - 3\vec{k}$  meet at a vertex, find the volume of the parallelepiped.

**Solution :**

Volume of parallelepiped :  $V = [\vec{a} \ \vec{b} \ \vec{c}]$

$$\begin{aligned}
 &= \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix} \\
 &= -3 \begin{vmatrix} 7 & -3 \\ -5 & -3 \end{vmatrix} - 7 \begin{vmatrix} -5 & -3 \\ 7 & -3 \end{vmatrix} + 5 \begin{vmatrix} -5 & 7 \\ 7 & -5 \end{vmatrix} \\
 &= -3(-21 - 15) - 7(15 + 21) + 5(25 - 49) \\
 &= -3(36) - 7(36) + 5(-24) \\
 &= 108 - 252 - 120 \\
 &= -264
 \end{aligned}$$

$\therefore$  Volume,  $V = 264$  cubic units.

3. Prove that the vectors  $3\vec{i} + 2\vec{j} - 2\vec{k}$ ,  $5\vec{i} - 3\vec{j} + 3\vec{k}$  and  $5\vec{i} - \vec{j} + \vec{k}$  are coplanar.

**Solution :**

$$\text{Let } \vec{a} = 3\vec{i} + 2\vec{j} - 2\vec{k}$$

$$\vec{b} = 5\vec{i} - 3\vec{j} + 3\vec{k}$$

$$\& \vec{c} = 5\vec{i} - \vec{j} + \vec{k}$$

$$\begin{aligned}
 \therefore [\vec{a} \ \vec{b} \ \vec{c}] &= \begin{vmatrix} 3 & 2 & -2 \\ 5 & -3 & 3 \\ 5 & -1 & 1 \end{vmatrix} \\
 &= 3 \begin{vmatrix} -3 & 3 \\ -1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 5 & 3 \\ 5 & 1 \end{vmatrix} - 2 \begin{vmatrix} 5 & -3 \\ 5 & -1 \end{vmatrix} \\
 &= 3(-3 + 3) - 2(5 - 15) - 2(-5 + 15) \\
 &= 3(0) - 2(-10) - 2(10) \\
 &= 20 - 20 = 0
 \end{aligned}$$

$\therefore \vec{a}, \vec{b}, \vec{c}$  are coplanar.

4. If the three vectors  $2\vec{i} - \vec{j} + \vec{k}$ ,  $\vec{i} + 2\vec{j} - 3\vec{k}$  and  $3\vec{i} + m\vec{j} + 5\vec{k}$  are coplanar, find the value of  $m$ .

**Solution :**

$$\text{Let } \vec{a} = 2\vec{i} - \vec{j} + \vec{k}$$

$$\vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}$$

$$\& \vec{c} = 3\vec{i} + m\vec{j} + 5\vec{k}$$

Since  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are coplanar then  $[\vec{a} \ \vec{b} \ \vec{c}] = 0$

$$\therefore \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & m & 5 \end{vmatrix} = 0$$

$$2 \begin{vmatrix} 2 & -3 \\ m & 5 \end{vmatrix} + 1 \begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 3 & m \end{vmatrix} = 0$$

$$2(10 + 3m) + 1(5 + 9) + 1(m - 6) = 0$$

$$20 + 6m + 14 + m - 6 = 0$$

$$7m + 28 = 0$$

$$7m = -28$$

$$\boxed{m = -4}$$

### PART - C

1. Find the magnitude of the moment about the point  $(1, -2, 3)$  of the force  $2\vec{i} + 3\vec{j} + 6\vec{k}$  whose line of action passes through the origin.

**Solution :**

$$\text{Given : } \vec{F} = 2\vec{i} + 3\vec{j} + 6\vec{k}$$

By data,

$$\vec{r} = (\text{Position Vector of } 0) - (\text{Position Vector of } A)$$

$$= (0, 0, 0) - (1, -2, 3)$$

$$= (-1, 2, -3)$$

$$\Rightarrow \vec{r} = -\vec{i} + 2\vec{j} - 3\vec{k}$$

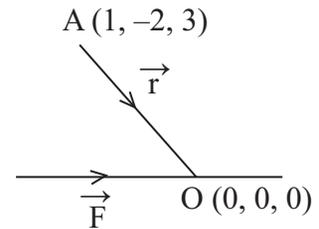
$\therefore$  Moment of force,

$$\vec{M} = \vec{r} \times \vec{F}$$

$$\begin{aligned}
 &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & -3 \\ 2 & 3 & 6 \end{vmatrix} \\
 &= \vec{i} \begin{vmatrix} 2 & -3 \\ 3 & 6 \end{vmatrix} - \vec{j} \begin{vmatrix} -1 & -3 \\ 2 & 6 \end{vmatrix} + \vec{k} \begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix} \\
 &= \vec{i} (12+9) - \vec{j} (-6+6) + \vec{k} (-3-4) \\
 &= \vec{i} (21) - \vec{j} (0) + \vec{k} (-7) \\
 &= 21\vec{i} - 7\vec{k} \\
 &\Rightarrow \boxed{\vec{M} = 7(3\vec{i} - \vec{k})}
 \end{aligned}$$

∴ Magnitude of moment,

$$\begin{aligned}
 |\vec{M}| &= 7\sqrt{(3)^2 + (-1)^2} \\
 &= 7\sqrt{9+1} \\
 \Rightarrow \vec{M} &= 7\sqrt{10} \text{ units}
 \end{aligned}$$

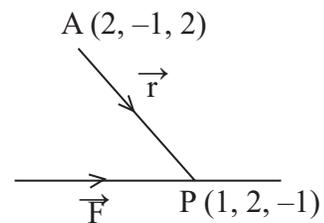


2. Find the moment of the force  $3\vec{i} + \vec{k}$  acting along the point  $\vec{i} + 2\vec{j} - \vec{k}$  about the point  $2\vec{i} + \vec{j} - 2\vec{k}$

**Solution :**

$$\begin{aligned}
 \text{Given :} \quad \vec{F} &= 3\vec{i} + \vec{k} \\
 \vec{OP} &= \vec{i} + 2\vec{j} - \vec{k} \\
 \& \quad \vec{OA} &= 2\vec{i} + \vec{j} - 2\vec{k}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } \vec{r} &= \vec{AP} = \vec{OP} - \vec{OA} \\
 &= (\vec{i} + 2\vec{j} - \vec{k}) - (2\vec{i} + \vec{j} - 2\vec{k}) \\
 &= \vec{i} + 2\vec{j} - \vec{k} - 2\vec{i} + \vec{j} - 2\vec{k} \\
 &\Rightarrow \boxed{\vec{r} = -\vec{i} + 3\vec{j} - 3\vec{k}}
 \end{aligned}$$



$$\text{Moment, } \vec{M} = \vec{r} \times \vec{F}$$

$$\begin{aligned}
 &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & -3 \\ 3 & 0 & 1 \end{vmatrix} \\
 &= \vec{i} \begin{vmatrix} 3 & -3 \\ 0 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} -1 & -3 \\ 3 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} -1 & 3 \\ 3 & 0 \end{vmatrix}
 \end{aligned}$$

$$\begin{aligned}
&= \vec{i} (3+0) - \vec{j} (-1+9) + \vec{k} (0-9) \\
&= \vec{i} (3) - \vec{j} (8) + \vec{k} (-9) \\
\Rightarrow \boxed{\vec{M} = 3\vec{i} - 8\vec{j} - 9\vec{k}} \\
\therefore |\vec{M}| &= \sqrt{(3)^2 + (-8)^2 + (-9)^2} = \sqrt{9+64+81} \Rightarrow \boxed{|\vec{M}| = \sqrt{154} \text{ units}}
\end{aligned}$$

3. Prove that  $[\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}] = 2[\vec{a} \ \vec{b} \ \vec{c}]$

**Solution :**

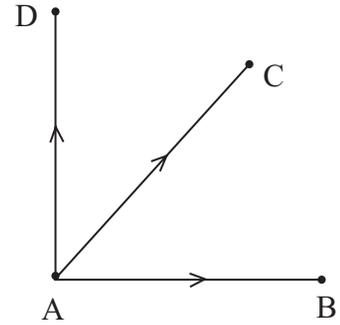
$$\begin{aligned}
\text{L.H.S: } &[\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}] \\
&= (\vec{a} + \vec{b}) \cdot \{(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})\} \\
&= (\vec{a} + \vec{b}) \cdot \{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}\} \\
&= (\vec{a} + \vec{b}) \cdot \{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{0} + \vec{c} \times \vec{a}\} \\
&= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{a}) + \vec{a} \cdot (\vec{c} \times \vec{a}) \\
&\quad + \vec{b} \cdot (\vec{b} \times \vec{c}) + \vec{b} \cdot (\vec{b} \times \vec{a}) + \vec{b} \cdot (\vec{c} \times \vec{a}) \\
&= [\vec{a} \ \vec{b} \ \vec{c}] + [\vec{a} \ \vec{b} \ \vec{a}] + [\vec{a} \ \vec{c} \ \vec{a}] \\
&\quad + [\vec{b} \ \vec{b} \ \vec{c}] + [\vec{b} \ \vec{b} \ \vec{a}] + [\vec{b} \ \vec{c} \ \vec{a}] \\
&= [\vec{a} \ \vec{b} \ \vec{c}] + 0 + 0 + 0 + 0 + [\vec{a} \ \vec{b} \ \vec{c}] \\
&= 2[\vec{a} \ \vec{b} \ \vec{c}] = \text{R.H.S.}
\end{aligned}$$

4. Prove that the points given by the vectors  $4\vec{i} + 5\vec{j} + \vec{k}$ ,  $-\vec{j} - \vec{k}$ ,  $3\vec{i} + 9\vec{j} + 4\vec{k}$  and  $-4\vec{i} + 4\vec{j} + 4\vec{k}$  are coplanar.

**Solution :**

$$\begin{aligned}
\text{Let } \vec{OA} &= 4\vec{i} + 5\vec{j} + \vec{k} \\
\vec{OB} &= -\vec{j} - \vec{k} \\
\vec{OC} &= 3\vec{i} + 9\vec{j} + 4\vec{k} \\
&\& \vec{OD} = -4\vec{i} + 4\vec{j} + 4\vec{k}
\end{aligned}$$

$$\begin{aligned}
 \text{Now, } \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\
 &= (-\vec{j} - \vec{k}) - (4\vec{i} + 5\vec{j} + \vec{k}) \\
 &= -\vec{j} - \vec{k} - 4\vec{i} - 5\vec{j} - \vec{k} \\
 \Rightarrow \overrightarrow{AB} &= -4\vec{i} - 6\vec{j} - 2\vec{k}
 \end{aligned}$$



$$\begin{aligned}
 \text{also, } \overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA} \\
 &= (3\vec{i} + 9\vec{j} + 4\vec{k}) - (4\vec{i} + 5\vec{j} + \vec{k}) \\
 &= 3\vec{i} + 9\vec{j} + 4\vec{k} - 4\vec{i} - 5\vec{j} - \vec{k} \\
 \Rightarrow \overrightarrow{AC} &= -\vec{i} + 4\vec{j} + 3\vec{k}
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \overrightarrow{AD} &= \overrightarrow{OD} - \overrightarrow{OA} \\
 &= (-4\vec{i} + 4\vec{j} + 4\vec{k}) - (4\vec{i} + 5\vec{j} + \vec{k}) \\
 &= -4\vec{i} + 4\vec{j} + 4\vec{k} - 4\vec{i} - 5\vec{j} - \vec{k} \\
 \Rightarrow \overrightarrow{AD} &= -8\vec{i} - \vec{j} + 3\vec{k}
 \end{aligned}$$

$$\begin{aligned}
 \therefore [\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] &= \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} \\
 &= -4 \begin{vmatrix} 4 & 3 \\ -1 & 3 \end{vmatrix} + 6 \begin{vmatrix} -1 & 3 \\ -8 & 3 \end{vmatrix} - 2 \begin{vmatrix} -1 & 4 \\ -8 & -1 \end{vmatrix} \\
 &= -4(12 + 3) + 6(-3 + 24) - 2(1 + 32) \\
 &= 4(15) + 6(21) - 2(33) \\
 &= -60 + 126 - 66 \\
 &= 0
 \end{aligned}$$

$\therefore$  The points given by the vectors are coplanar.

### 3.3 VECTOR TRIPLE PRODUCT AND PRODUCT OF MORE VECTORS

#### Definition : Vector Triple Product

Let  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  be any three vectors then the product  $\vec{a} \times (\vec{b} \times \vec{c})$  &  $(\vec{a} \times \vec{b}) \times \vec{c}$  are called vector triple product of  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$

#### Result :

$$(i) \vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

$$(ii) (\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a}$$

$$(iii) \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

Note : The vector triple product  $(\vec{a} \times \vec{b}) \times \vec{c}$  is perpendicular to  $\vec{c}$  and lies in the plane which contain  $\vec{a}$  and  $\vec{b}$ .

#### PRODUCT OF FOUR VECTORS

#### Definition : Scalar product of four Vectors

If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$ ,  $\vec{d}$  are four vectors then the scalar product of these four factors is defined as  $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

#### Result : Determinant form of $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

#### Proof :

$$\begin{aligned} (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) &= (\vec{a} \times \vec{b}) \cdot \vec{x}, \text{ where } \vec{x} = \vec{c} \times \vec{d} \\ &= \vec{a} \cdot (\vec{b} \times \vec{x}) \\ &= \vec{a} \cdot [\vec{b} \times (\vec{c} \times \vec{d})] \\ &= \vec{a} \cdot [(\vec{b} \cdot \vec{d}) \vec{c} - (\vec{b} \cdot \vec{c}) \vec{d}] \\ &= (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) \\ \Rightarrow (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) &= \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix} \end{aligned}$$

**Definition : Vector product of four vectors**

If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  and  $\vec{d}$  are four vectors, then the vector product of the vectors  $(\vec{a} \times \vec{b})$  and  $(\vec{c} \times \vec{d})$  is defined as vector product of four vectors and is denoted by  $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ .

**Results :**

1. If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$ ,  $\vec{d}$  are four vectors then

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a} \ \vec{b} \ \vec{d}] \vec{c} - [\vec{a} \ \vec{b} \ \vec{c}] \vec{d}$$

**Proof :**

$$\begin{aligned} \text{L.H.S.} &= (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) \\ &= \vec{x} \times (\vec{c} \times \vec{d}) \text{ where } \vec{x} = \vec{a} \times \vec{b} \\ &= (\vec{x} \cdot \vec{d}) \vec{c} - (\vec{x} \cdot \vec{c}) \vec{d} \\ &= [(\vec{a} \times \vec{b}) \cdot \vec{d}] \vec{c} - [(\vec{a} \times \vec{b}) \cdot \vec{c}] \vec{d} \\ &= [\vec{a} \ \vec{b} \ \vec{d}] \vec{c} - [\vec{a} \ \vec{b} \ \vec{c}] \vec{d} = \text{R.H.S.} \end{aligned}$$

2. If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$ ,  $\vec{d}$  are coplanar then  $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$ .

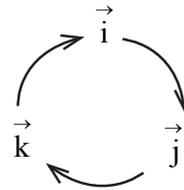
3.  $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a} \ \vec{c} \ \vec{d}] \vec{b} - [\vec{b} \ \vec{c} \ \vec{d}] \vec{a}$

**WORKED EXAMPLES****PART - A**

1. Find the value of  $\vec{i} \times (\vec{j} \times \vec{k})$

**Solution :**

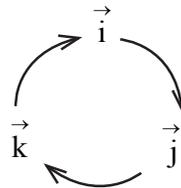
$$\begin{aligned} \vec{i} \times (\vec{j} \times \vec{k}) &= \vec{i} \times \vec{i} \\ &= \vec{0} \end{aligned}$$



2. Find the value of  $\vec{k} \times (\vec{j} \times \vec{k})$

**Solution :**

$$\begin{aligned} \vec{k} \times (\vec{j} \times \vec{k}) &= \vec{k} \times \vec{i} \\ &= \vec{j} \end{aligned}$$



3. Find the value of  $(\vec{a} \times \vec{m}) \cdot (\vec{b} \times \vec{n})$

**Solution :**

$$\begin{aligned} (\vec{a} \times \vec{m}) \cdot (\vec{b} \times \vec{n}) &= \begin{vmatrix} \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{n} \\ \vec{m} \cdot \vec{b} & \vec{m} \cdot \vec{n} \end{vmatrix} \\ &= (\vec{a} \cdot \vec{b})(\vec{m} \cdot \vec{n}) - (\vec{m} \cdot \vec{b})(\vec{a} \cdot \vec{n}) \end{aligned}$$

4. If  $[\vec{a} \vec{b} \vec{d}] = 5$ ,  $[\vec{a} \vec{b} \vec{c}] = 2$ ,  $\vec{c} = \vec{i} - \vec{j} - \vec{k}$  and  $\vec{d} = 2\vec{i} + 3\vec{j} - 4\vec{k}$  and  $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

**Solution :**

$$\begin{aligned}(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) &= [\vec{a} \vec{b} \vec{d}] \vec{c} - [\vec{a} \vec{b} \vec{c}] \vec{d} \\ &= 5(\vec{i} - \vec{j} - \vec{k}) - 2(2\vec{i} + 3\vec{j} - 4\vec{k}) \\ &= 5\vec{i} - 5\vec{j} - 5\vec{k} - 4\vec{i} - 6\vec{j} + 8\vec{k}\end{aligned}$$

$$\Rightarrow (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{i} - 11\vec{j} + 3\vec{k}$$

5. If  $\vec{a} \times \vec{b} = 7\vec{i} - 3\vec{j} + 4\vec{k}$  &  $\vec{c} \times \vec{d} = \vec{i} + 3\vec{j} - 2\vec{k}$  find  $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

**Solution :**

$$\begin{aligned}(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) &= 7\vec{i} - 3\vec{j} + 4\vec{k} \cdot (\vec{i} + 3\vec{j} - 2\vec{k}) \\ &= (7 \times 1) + (-3 \times 3) + (4 \times -2) \\ &= 7 - 9 - 8 \\ &= -10\end{aligned}$$

### PART - B

1. If  $\vec{a} = \vec{i} + \vec{j}$ ,  $\vec{b} = \vec{j} + \vec{k}$  &  $\vec{c} = \vec{k} + \vec{i}$  find  $\vec{a} \times (\vec{b} \times \vec{c})$

**Solution :**

$$\begin{aligned}\vec{b} \times \vec{c} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \\ &= \vec{i}(1-0) - \vec{j}(0-1) + \vec{k}(0-1) \\ &= \vec{i}(1) - \vec{j}(-1) + \vec{k}(-1) \\ \Rightarrow \vec{b} \times \vec{c} &= \vec{i} + \vec{j} - \vec{k} \\ \therefore \vec{a} \times (\vec{b} \times \vec{c}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} \\ &= \vec{i}(-1-0) - \vec{j}(-1-0) + \vec{k}(1-1) \\ &= \vec{i}(-1) - \vec{j}(-1) + \vec{k}(0) \\ \Rightarrow \vec{a} \times (\vec{b} \times \vec{c}) &= -\vec{i} + \vec{j}\end{aligned}$$

2. For any vector  $\vec{a}$  prove that  $\vec{i} \times (\vec{a} \times \vec{i}) + \vec{j} \times (\vec{a} \times \vec{j}) + \vec{k} \times (\vec{a} \times \vec{k}) = 2\vec{a}$ .

**Solution :**

We have for any vectors  $\vec{a}$  then  $(\vec{a} \cdot \vec{i}) \vec{i} + (\vec{a} \cdot \vec{j}) \vec{j} + (\vec{a} \cdot \vec{k}) \vec{k} = \vec{a}$  ——— (1)

Now,

$$\begin{aligned}\vec{i} \times (\vec{a} \times \vec{i}) &= (\vec{i} \cdot \vec{i}) \vec{a} - (\vec{i} \cdot \vec{a}) \vec{i} \\ &= \vec{a} - (\vec{a} \cdot \vec{i}) \vec{i}\end{aligned}$$

$$\text{Similarly, } \vec{j} \times (\vec{a} \times \vec{j}) = \vec{a} - (\vec{a} \cdot \vec{j}) \vec{j}$$

$$\& \vec{k} \times (\vec{a} \times \vec{k}) = \vec{a} - (\vec{a} \cdot \vec{k}) \vec{k}$$

$$\begin{aligned}\text{L.H.S.} &= \vec{i} \times (\vec{a} \times \vec{i}) + \vec{j} \times (\vec{a} \times \vec{j}) + \vec{k} \times (\vec{a} \times \vec{k}) \\ &= \vec{a} - (\vec{a} \cdot \vec{i}) \vec{i} + \vec{a} - (\vec{a} \cdot \vec{j}) \vec{j} + \vec{a} - (\vec{a} \cdot \vec{k}) \vec{k} \\ &= 3\vec{a} - [(\vec{a} \cdot \vec{i}) \vec{i} + (\vec{a} \cdot \vec{j}) \vec{j} + (\vec{a} \cdot \vec{k}) \vec{k}] \\ &= 3\vec{a} - \vec{a} = 2\vec{a} \text{ (R.H.S.)}\end{aligned}$$

### PART - C

1. If  $\vec{a} = \vec{i} - \vec{j} + \vec{k}$ ,  $\vec{b} = \vec{i} - 2\vec{j}$ ,  $\vec{c} = 2\vec{i} - \vec{j} + \vec{k}$  prove that  $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ .

**Solution :**

L.H.S.

$$\begin{aligned}\text{Now, } \vec{b} \times \vec{c} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 0 \\ 2 & -1 & 1 \end{vmatrix} \\ &= \vec{i} \begin{vmatrix} -2 & 0 \\ -1 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} \\ &= \vec{i} (-2+0) - \vec{j} (1-0) + \vec{k} (-1+4) \\ &= \vec{i} (-2) - \vec{j} (1) + \vec{k} (3)\end{aligned}$$

$$\Rightarrow \boxed{\vec{b} \times \vec{c} = -2\vec{i} - \vec{j} + 3\vec{k}}$$

$$\therefore \vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ -2 & -1 & 3 \end{vmatrix}$$

$$\begin{aligned}
&= \vec{i} \begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & -1 \\ -2 & -1 \end{vmatrix} \\
&= \vec{i} (-3+1) - \vec{j} (3+2) + \vec{k} (-1-2) \\
&= \vec{i} (-2) - \vec{j} (5) + \vec{k} (-3) \\
\Rightarrow \boxed{\vec{a} \times (\vec{b} \times \vec{c}) = -2\vec{i} - 5\vec{j} - 3\vec{k}} \quad \text{———— (1)}
\end{aligned}$$

R.H.S.

$$\begin{aligned}
\text{Now, } \vec{a} \cdot \vec{c} &= (\vec{i} - \vec{j} + \vec{k}) \cdot (2\vec{i} - \vec{j} + \vec{k}) \\
&= (1 \times 2) + (-1 \times -1) + (1 \times 1) \\
&= 2 + 1 + 1 = 4
\end{aligned}$$

$$\begin{aligned}
\text{also, } \vec{a} \cdot \vec{b} &= (\vec{i} - \vec{j} + \vec{k}) \cdot (\vec{i} - 2\vec{j}) \\
&= (1 \times 1) + (-1 \times -2) \\
&= 1 + 2 = 3
\end{aligned}$$

$$\begin{aligned}
\therefore (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} &= 4[\vec{i} - 2\vec{j}] - 3[2\vec{i} - \vec{j} + \vec{k}] \\
&= 4\vec{i} - 8\vec{j} - 6\vec{i} + 3\vec{j} - 3\vec{k}
\end{aligned}$$

$$\Rightarrow \boxed{(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = -2\vec{i} - 5\vec{j} - 3\vec{k}} \quad \text{———— (2)}$$

From (1) & (2) we conclude that,  $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

2. If  $\vec{a} = \vec{i} - \vec{j} + \vec{k}$ ,  $\vec{b} = -\vec{i} + 2\vec{j} - \vec{k}$ ,  $\vec{c} = \vec{i} + 2\vec{j}$  and  $\vec{d} = \vec{i} - \vec{j} - 3\vec{k}$  find  $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

**Solution :**

$$\text{Given : } \vec{a} = \vec{i} - \vec{j} + \vec{k} \quad \vec{b} = -\vec{i} + 2\vec{j} - \vec{k} \quad \vec{c} = \vec{i} + 2\vec{j} \quad \vec{d} = \vec{i} - \vec{j} - 3\vec{k}$$

Now,

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ -1 & 2 & -1 \end{vmatrix} \\
&= \vec{i} \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} \\
&= \vec{i} (1-2) - \vec{j} (-1+1) + \vec{k} (2-1) \\
&= \vec{i} (-1) - \vec{j} (0) + \vec{k} (1)
\end{aligned}$$

$$\Rightarrow \boxed{\vec{a} \times \vec{b} = -\vec{i} + \vec{k}}$$

$$\begin{aligned} \text{also, } \vec{c} \times \vec{d} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 1 & -1 & -3 \end{vmatrix} \\ &= \vec{i} \begin{vmatrix} 2 & 0 \\ -1 & -3 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 0 \\ 1 & -3 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} \\ &= \vec{i} (-6+0) - \vec{j} (-3-0) + \vec{k} (-1-2) \\ &= \vec{i} (-6) - \vec{j} (-3) + \vec{k} (-3) \end{aligned}$$

$$\Rightarrow \boxed{\vec{c} \times \vec{d} = -6\vec{i} + 3\vec{j} - 3\vec{k}}$$

$$\begin{aligned} \therefore (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) &= (-\vec{i} + \vec{k}) \cdot (6\vec{i} + 3\vec{j} - 3\vec{k}) \\ &= (-1 \times 6) + (1 \times -3) \\ &= -6 - 3 \end{aligned}$$

$$\Rightarrow \boxed{(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = -9}$$

#### Alternate method :

$$\begin{aligned} \vec{a} \cdot \vec{c} &= (\vec{i} - \vec{j} + \vec{k}) \cdot (\vec{i} + 2\vec{j}) \\ &= (1 \times 1) + (-1 \times 2) + (1 \times 0) \\ &= 1 - 2 + 0 \end{aligned}$$

$$\Rightarrow \boxed{\vec{a} \cdot \vec{c} = -1}$$

$$\begin{aligned} \vec{a} \cdot \vec{d} &= (\vec{i} - \vec{j} + \vec{k}) \cdot (\vec{i} - \vec{j} - 3\vec{k}) \\ &= (1 \times 1) + (-1 \times -1) + (1 \times -3) \\ &= 1 + 1 - 3 \end{aligned}$$

$$\Rightarrow \boxed{\vec{a} \cdot \vec{d} = -1}$$

$$\begin{aligned} \text{also, } \vec{b} \cdot \vec{c} &= (-\vec{i} + 2\vec{j} - \vec{k}) \cdot (\vec{i} + 2\vec{j}) \\ &= (-1 \times 1) + (2 \times 2) + (-1 \times 0) \\ &= -1 + 4 + 0 \end{aligned}$$

$$\Rightarrow \boxed{\vec{b} \cdot \vec{c} = 3}$$

$$\begin{aligned} \text{and } \vec{b} \cdot \vec{d} &= (-\vec{i} + 2\vec{j} - \vec{k}) \cdot (\vec{i} - \vec{j} - 3\vec{k}) \\ &= (-1 \times 1) + (2 \times -1) + (-1 \times -3) \\ &= -1 - 2 + 3 \end{aligned}$$

$$\Rightarrow \boxed{\vec{b} \cdot \vec{d} = 0}$$

$$\begin{aligned} \therefore (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) &= \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix} \\ &= \begin{vmatrix} -1 & -1 \\ 3 & 0 \end{vmatrix} \\ &= 0 + 3 \end{aligned}$$

$$\Rightarrow (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = 3$$

3. If  $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ ,  $\vec{b} = \vec{i} - \vec{j} - \vec{k}$ ,  $\vec{c} = -\vec{i} + \vec{j} + 2\vec{k}$ ,  $\vec{d} = 2\vec{i} + \vec{j}$  find  $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

**Solution :**

$$\begin{aligned} \text{Given : } \vec{a} &= \vec{i} + \vec{j} + \vec{k} & \vec{b} &= \vec{i} - \vec{j} - \vec{k} \\ \vec{c} &= -\vec{i} + \vec{j} + 2\vec{k} & \vec{d} &= 2\vec{i} + \vec{j} \end{aligned}$$

$$\begin{aligned} \text{Now, } \vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix} \\ &= \vec{i} \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\ &= \vec{i} (-1 + 1) - \vec{j} (-1 - 1) + \vec{k} (-1 - 1) \\ &= \vec{i} (0) - \vec{j} (-2) + \vec{k} (-2) \end{aligned}$$

$$\Rightarrow \boxed{\vec{a} \times \vec{b} = 2\vec{j} - 2\vec{k}}$$

$$\begin{aligned} \text{also, } \vec{c} \times \vec{d} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ 2 & 1 & 0 \end{vmatrix} \\ &= \vec{i} \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} \\ &= \vec{i} (0 - 2) - \vec{j} (0 - 4) + \vec{k} (-1 - 2) \\ &= \vec{i} (-2) - \vec{j} (-4) + \vec{k} (-3) \end{aligned}$$

$$\Rightarrow \boxed{\vec{c} \times \vec{d} = -2\vec{i} + 4\vec{j} - 3\vec{k}}$$

$$\therefore (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & -2 \\ -2 & 4 & -3 \end{vmatrix}$$

$$\begin{aligned}
&= \vec{i} \begin{vmatrix} 2 & -2 \\ 4 & -3 \end{vmatrix} - \vec{j} \begin{vmatrix} 0 & -2 \\ -2 & -3 \end{vmatrix} + \vec{k} \begin{vmatrix} 0 & 2 \\ -2 & 4 \end{vmatrix} \\
&= \vec{i} (-6+8) - \vec{j} (0-4) + \vec{k} (0+4) \\
&= \vec{i} (2) - \vec{j} (-4) + \vec{k} (4)
\end{aligned}$$

$$\Rightarrow \boxed{(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = 2\vec{i} + 4\vec{j} + 4\vec{k}}$$

Alternate method :

We have  $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a} \ \vec{b} \ \vec{d}] \vec{c} - [\vec{a} \ \vec{b} \ \vec{c}] \vec{d}$

$$\begin{aligned}
[\vec{a} \ \vec{b} \ \vec{d}] &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 1 & 0 \end{vmatrix} \\
&= 1 \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \\
&= 1(0+1) - 1(0+2) + 1(1+2) \\
&= 1(1) - 1(2) + 1(3) = 1 - 2 + 3
\end{aligned}$$

$$\Rightarrow \boxed{[\vec{a} \ \vec{b} \ \vec{d}] = 2}$$

$$\begin{aligned}
\text{also, } [\vec{a} \ \vec{b} \ \vec{c}] &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 2 \end{vmatrix} \\
&= 1 \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \\
&= 1(-2+1) - 1(2-1) + 1(1-1) \\
&= 1(-1) - 1(1) + 1(0) \\
&= -1 - 1
\end{aligned}$$

$$\Rightarrow \boxed{[\vec{a} \ \vec{b} \ \vec{c}] = -2}$$

$$\begin{aligned}
\therefore (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) &= [\vec{a} \ \vec{b} \ \vec{d}] \vec{c} - [\vec{a} \ \vec{b} \ \vec{c}] \vec{d} = 2(-\vec{i} + \vec{j} + 2\vec{k}) + 2(2\vec{i} + \vec{j}) \\
&= -2\vec{i} + 2\vec{j} + 4\vec{k} + 4\vec{i} + 2\vec{j}
\end{aligned}$$

$$\Rightarrow \boxed{(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = 2\vec{i} + 4\vec{j} + 4\vec{k}}$$

4. Prove that  $[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}] = [\vec{a} \ \vec{b} \ \vec{c}]^2$

**Solution :**

$$\begin{aligned}
 \text{L.H.S.} &: [\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}] \\
 &= \{(\vec{a} \times \vec{b}) \cdot [(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})]\} \\
 &= \{(\vec{a} \times \vec{b}) \cdot \{[\vec{b} \ \vec{c} \ \vec{a}] \vec{c} - [\vec{b} \ \vec{c} \ \vec{c}] \vec{a}\}\} \\
 &= \{(\vec{a} \times \vec{b}) \cdot \{[\vec{b} \ \vec{c} \ \vec{a}] \vec{c} - 0 \cdot \vec{a}\}\} \\
 &= \{[(\vec{a} \times \vec{b}) \cdot \vec{c}] [\vec{b} \ \vec{c} \ \vec{a}]\} \\
 &= [\vec{a} \ \vec{b} \ \vec{c}] [\vec{a} \ \vec{b} \ \vec{c}] \\
 &= [\vec{a} \ \vec{b} \ \vec{c}]^2 = \text{R.H.S.} \qquad \left[ \because [\vec{b} \ \vec{c} \ \vec{a}] = [\vec{a} \ \vec{b} \ \vec{c}] \right] \\
 &\Rightarrow \text{L.H.S.} = \text{R.H.S.}
 \end{aligned}$$

## EXERCISE

### PART - A

1. Prove that  $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$ .
2. Find  $\vec{a} \times \vec{b}$  if  $\vec{a} = 2\vec{i} + \vec{k}$ ,  $\vec{b} = \vec{i} + \vec{j} + \vec{k}$ .
3. Find  $\vec{a} \times \vec{b}$  if  $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$  and  $\vec{b} = \vec{i} - \vec{j} - \vec{k}$ .
4. If  $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$ ,  $\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$  find  $\vec{a} \times \vec{b}$ .
5. If  $|\vec{a}| = 2$ ,  $|\vec{b}| = 7$  and  $|\vec{a} \times \vec{b}| = 7\sqrt{3}$ , find the angle between  $\vec{a}$  and  $\vec{b}$ .
6. If  $|\vec{a}| = 3$ ,  $|\vec{b}| = 4$  and  $|\vec{a} \times \vec{b}| = 6$  find angle between  $\vec{a}$  and  $\vec{b}$ .
7. Find the angle between the vectors  $\vec{a}$  and  $\vec{b}$  if  $|\vec{a} \times \vec{b}| = \vec{a} \cdot \vec{b}$ .
8. Find the value of  $[\vec{j} \ \vec{k} \ \vec{i}]$ .
9. Find the value of  $[\vec{i} - \vec{j}, \vec{j} - \vec{k}, \vec{k} - \vec{i}]$
10. If  $\vec{F} = 2\vec{i} + 3\vec{j} + 6\vec{k}$  and  $\vec{r} = -\vec{i} + 2\vec{j} - 3\vec{k}$  find the moment of the force.
11. Find the value of
  - (i)  $\vec{i} \times (\vec{j} \times \vec{k})$
  - (ii)  $\vec{k} \times (\vec{k} \times \vec{i})$
  - (iii)  $\vec{i} \times (\vec{j} \times \vec{i})$
  - (iv)  $\vec{k} \times (\vec{j} \times \vec{k})$
12. If  $[\vec{a} \ \vec{b} \ \vec{d}] = 4$ ,  $[\vec{a} \ \vec{b} \ \vec{c}] = -2$ ,  $\vec{c} = \vec{i} - 2\vec{j}$  and  $\vec{d} = 3\vec{j} + \vec{k}$  find  $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ .
13. If  $[\vec{a} \ \vec{c} \ \vec{d}] = 1$ ,  $[\vec{b} \ \vec{c} \ \vec{d}] = 3$ ,  $\vec{a} = \vec{i} + \vec{j} + \vec{k}$  &  $\vec{b} = \vec{i} - \vec{j} - \vec{k}$  find  $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ .

**PART - B**

- Find the area of the parallelogram whose adjacent sides are  $\vec{i} + \vec{j} + \vec{k}$  and  $\vec{i} + 2\vec{j} - 3\vec{k}$
- Find the area of the parallelogram whose adjacent sides are  $2\vec{i} + 3\vec{j} + 6\vec{k}$  and  $3\vec{i} - 6\vec{j} + 2\vec{k}$
- Find the area of the triangle whose adjacent sides are  $\vec{i} + \vec{j} - 2\vec{k}$  and  $2\vec{i} - \vec{j} - \vec{k}$ .
- Find area of the triangle whose adjacent sides are  $2\vec{i} - \vec{j} + \vec{k}$  and  $\vec{i} + 2\vec{j} + 3\vec{k}$ .
- Find the volume of the parallelepiped whose edges are
  - $4\vec{i} - 8\vec{j} + \vec{k}, 2\vec{i} - \vec{j} - 2\vec{k}, 3\vec{i} - 4\vec{j} + 12\vec{k}$
  - $2\vec{i} - 3\vec{j} + 4\vec{k}, \vec{i} + 2\vec{j} - \vec{k}, 3\vec{i} - \vec{j} + 2\vec{k}$
- Show that the following vectors are coplanar.
  - $2\vec{i} + \vec{j} + \vec{k}, 3\vec{i} + 4\vec{j} + \vec{k}, \vec{i} - 2\vec{j} + \vec{k}$
  - $2\vec{i} - 3\vec{j} + 5\vec{k}, \vec{i} + 2\vec{j} - \vec{k}, 3\vec{i} - \vec{j} + 4\vec{k}$
  - $3\vec{i} + 2\vec{j} - 5\vec{k}, \vec{i} - 2\vec{j} - 3\vec{k}, 3\vec{i} + 10\vec{j} + 19\vec{k}$
- Find the value of 'm' so that the vectors  $2\vec{i} + \vec{j} - 2\vec{k}, \vec{i} + \vec{j} + 3\vec{k}$  and  $m\vec{i} + \vec{j}$  are coplanar.
- If  $\vec{a} = -4\vec{i} - 6\vec{j} - 2\vec{k}, \vec{b} = -\vec{i} + 4\vec{j} + 3\vec{k}$  &  $\vec{c} = -8\vec{i} - \vec{j} + 3\vec{k}$  find  $[\vec{a} \vec{b} \vec{c}]$ .
- If  $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}, \vec{b} = -2\vec{i} + 5\vec{k}$  &  $\vec{c} = \vec{j} - 3\vec{k}$  find  $\vec{a} \times (\vec{b} \times \vec{c})$ .
- If  $\vec{a} = 3\vec{i} + 2\vec{j} - 4\vec{k}, \vec{b} = 5\vec{i} - 3\vec{j} + 6\vec{k}$  &  $\vec{c} = 5\vec{i} - \vec{j} + 2\vec{k}$  find  $(\vec{a} \times \vec{b}) \times \vec{c}$ .

**PART - C**

- Find the unit vector perpendicular to each of the vectors  $2\vec{i} - \vec{j} + 2\vec{k}$  and  $10\vec{i} - 2\vec{j} + 7\vec{k}$ . Find also the sine of the angle between them.
- Find the unit vector perpendicular to the vectors  $-\vec{i} + \vec{j} + 2\vec{k}$  and  $-4\vec{i} + 3\vec{j} + 2\vec{k}$ . Also find the sine of the angle between them.
- Find the unit vector perpendicular to each of the vectors  $3\vec{i} + \vec{j} + 2\vec{k}$  and  $2\vec{i} - 2\vec{j} + 4\vec{k}$ . Also find the sine of the angle between them.
- Find the area of the triangle formed by the points whose position vectors are
  - $\vec{i} + 3\vec{j} + 2\vec{k}, 2\vec{i} - \vec{j} + \vec{k}, -\vec{i} + 2\vec{j} + 3\vec{k}$
  - $(3, -1, 2), (1, -1, -3), (4, -3, 1)$
- Find the area of the parallelogram whose diagonals are represented by
  - $3\vec{i} + \vec{j} - 2\vec{k}$  and  $\vec{i} - 3\vec{j} + 4\vec{k}$
  - $2\vec{i} + \vec{j} - \vec{k}$  and  $3\vec{i} - 4\vec{j} + \vec{k}$

6. Find the moment about the point  $\vec{i} + 2\vec{j} - \vec{k}$  of the force represented by  $3\vec{i} + \vec{k}$  acting through the point  $2\vec{i} - \vec{j} - 3\vec{k}$
7. Show that torque about the point A (3, -1, 3) of the force  $4\vec{i} + 2\vec{j} + \vec{k}$  through the point B (5, 2, 4) is  $\vec{i} + 2\vec{j} + 8\vec{k}$
8. Find the moment of the force  $3\vec{i} + \vec{j} + 2\vec{k}$  acting through the point  $\vec{i} - \vec{j} + 2\vec{k}$  about the point  $2\vec{i} - \vec{j} + 3\vec{k}$ .
9. Show that the points given by the position vectors are coplanar.
- (i) (1, 3, 1), (1, 1, -1), (-1, 1, 1), (2, 2, -1)
- (ii) (1, 2, 2), (3, -1, 2), (-2, 3, 2), (6, -4, 2)
10. If  $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$ ,  $\vec{b} = -2\vec{i} + 5\vec{k}$  &  $\vec{c} = \vec{j} - 3\vec{k}$  verify that  $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ .
11. If  $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ ,  $\vec{b} = 3\vec{i} - 2\vec{j} + \vec{k}$  &  $\vec{c} = 2\vec{i} - \vec{j} - 4\vec{k}$  verify that  $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ .
12. If  $\vec{a} = 3\vec{i} - 4\vec{j} + 5\vec{k}$ ,  $\vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}$  &  $\vec{c} = 2\vec{i} - \vec{j} + \vec{k}$  show that  $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{b})\vec{c}$ .
13. If  $\vec{a} = 2\vec{i} + 3\vec{j} - 5\vec{k}$ ,  $\vec{b} = -\vec{i} + \vec{j} + \vec{k}$  &  $\vec{c} = 4\vec{i} - 2\vec{j} + 3\vec{k}$  show that  $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{b})\vec{c}$ .
14. If  $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ ,  $\vec{b} = 2\vec{i} + \vec{k}$ ,  $\vec{c} = 2\vec{i} + \vec{j} + \vec{k}$  &  $\vec{d} = \vec{i} + \vec{j} + 2\vec{k}$  find  $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$
15. If  $\vec{a} = \vec{i} - \vec{j} + \vec{k}$ ,  $\vec{b} = 2\vec{i} + 3\vec{j} - 5\vec{k}$ ,  $\vec{c} = 2\vec{i} + 3\vec{j} - \vec{k}$  &  $\vec{d} = \vec{i} + \vec{j} - \vec{k}$  find  $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$
16. If  $\vec{a} = \vec{i} + \vec{k}$ ,  $\vec{b} = \vec{i} + \vec{j}$ ,  $\vec{c} = \vec{i} - \vec{k}$  &  $\vec{d} = 2\vec{i} - \vec{j} + 3\vec{k}$  find  $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$
17. If  $\vec{a} = 3\vec{i} + 4\vec{j} + 2\vec{k}$ ,  $\vec{b} = \vec{i} + 2\vec{j} + 3\vec{k}$ ,  $\vec{c} = 4\vec{i} + 2\vec{j} + 5\vec{k}$  &  $\vec{d} = 4\vec{i} + 3\vec{j} + 7\vec{k}$  find  $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ .
18. If  $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ ,  $\vec{b} = 2\vec{i} + \vec{k}$ ,  $\vec{c} = 2\vec{i} + \vec{j} + \vec{k}$  &  $\vec{d} = \vec{i} + \vec{j} + 2\vec{k}$  verify that  $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a} \ \vec{b} \ \vec{d}]\vec{c} - [\vec{a} \ \vec{b} \ \vec{c}]\vec{d}$
19. If  $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ ,  $\vec{b} = \vec{i} - \vec{j} - \vec{k}$ ,  $\vec{c} = -\vec{i} - \vec{j} + 2\vec{k}$  &  $\vec{d} = 2\vec{i} + \vec{j}$  verify that  $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a} \ \vec{b} \ \vec{d}]\vec{c} - [\vec{a} \ \vec{b} \ \vec{c}]\vec{d}$ .

**ANSWERS****PART - A**

- 2)  $-\vec{i}-\vec{j}+2\vec{k}$       3)  $\vec{i}+4\vec{j}-3\vec{k}$       4)  $-3\vec{i}+5\vec{j}+11\vec{k}$       5)  $60^\circ$   
 6)  $30^\circ$       7)  $45^\circ$       8) 1      9) 0      10)  $21\vec{i}-7\vec{k}$   
 11) (i)  $\vec{0}$       (ii)  $-\vec{i}$       (iii)  $-\vec{j}$       (iv)  $-\vec{j}$   
 12)  $4\vec{i}-2\vec{j}+2\vec{k}$       13)  $-2\vec{i}-4\vec{j}-4\vec{k}$

**PART - B**

- 1)  $A = \sqrt{42}$  sq. units      2)  $A = \frac{49}{2}$  sq. units      3)  $A = 3\sqrt{3}$  sq. units  
 4)  $A = 5\sqrt{3}$  sq. units      5) (i) 155 cubic units      (ii) 7 cubic units  
 7)  $m = \frac{8}{5}$       8) 0      9)  $-12\vec{i}+9\vec{j}+3\vec{k}$       10)  $-95\vec{i}-95\vec{j}+190\vec{k}$

**PART - C**

- 1)  $\hat{n} = \frac{-\vec{i}+2\vec{j}+2\vec{k}}{3}$  ;  $\sin\theta = \frac{3}{\sqrt{153}}$       2)  $\hat{n} = \frac{-4\vec{i}-6\vec{j}+\vec{k}}{\sqrt{53}}$  ;  $\sin\theta = \frac{\sqrt{53}}{\sqrt{6}\sqrt{29}}$   
 3)  $\hat{n} = \frac{\vec{i}-\vec{j}-\vec{k}}{\sqrt{3}}$  ;  $\sin\theta = \frac{4\sqrt{3}}{\sqrt{14}\sqrt{3}}$       4) (i)  $A = \frac{3\sqrt{14}}{2}$  sq. units      (ii)  $A = \frac{\sqrt{165}}{2}$  sq. units  
 5) (i)  $A = 5\sqrt{3}$  sq. units      (ii)  $A = \frac{\sqrt{155}}{2}$       6)  $\vec{m} = -3\vec{i}-7\vec{j}+9\vec{k}$  ;  $|\vec{m}| = \sqrt{139}$  units  
 7)  $\vec{m} = -\vec{i}-\vec{j}-\vec{k}$  ;  $|\vec{m}| = \sqrt{3}$  units      14) -4      15) -2  
 16)  $4\vec{i}-2\vec{j}+6\vec{k}$       17)  $-12\vec{i}-34\vec{j}-71\vec{k}$

# UNIT – IV

## INTEGRAL CALCULUS-I

### 4.1 Introduction:

Definition of integration – Integral values using reverse process of differentiation – Integration using decomposition method. Simple problems.

### 4.2 Integration by Substitution:

Integrals of the form  $\int [f(x)]^n f'(x) dx$ ,  $n \neq -1$ ,  $\int \frac{f'(x) dx}{f(x)}$ , and  $\int [F(f(x))] f'(x) dx$ , Simple problems.

### 4.3 Standard Integrals

Integrals of the form  $\int \frac{dx}{a^2 \pm x^2}$ ,  $\frac{dx}{x^2 - a^2}$  and  $\int \frac{dx}{\sqrt{a^2 - x^2}}$ , Simple problems.

## 4.1 INTRODUCTION

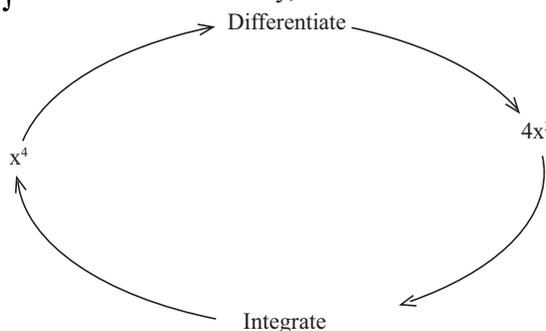
Sir Sardar Vallabhai Patel, called the Iron Man of India integrated several princely states together while forming our country Indian Nation after independence. Like that in Maths while finding area under a curve through integration, the area under the curve is divided into smaller rectangles and then integrating (i.e) summing of all the area of rectangles together. So, integration means of summation of very minute things of the same kind.

### Integration as the reverse of differentiation:

Integration can also be introduced in another way, called integration as the reverse of differentiation.

### Differentiation in reverse:

Suppose we differentiate the function  $y = x^4$ , then we have  $\frac{5}{4}$ . Now, we say that integral of  $4x^3$  is  $x^4$  and we write this as  $\int 4x^3 dx = x^4$ . Pictorially, we can think of this as follows:



Suppose we differentiate the functions,  $y = x^4 + 5$ ,  $y = x^4 - 23$ ,  $y = x^4 + 100$ , then also we get  $\frac{dy}{dx} = 4x^3$ . Now, what could we say about the integral of  $4x^3$ . Can we say that it is  $x^4 + 5$  (or)  $x^4 - 23$  (or)  $x^4 + 100$ ? So, in general we say that  $\int 4x^3 dx = x^4 + c$  where  $c$  is called constant of integration.

The symbol for integration is  $\int$ , known as integral sign. Along with the integral sign there is a term  $dx$  which must always be written and which indicates the name of the variable involved, in this case 'x'. Technically integrals of this sort are called indefinite Integrals.

### List of Formulae:

Sl.No.	Integration	Reverse Process of Differentiation
1.	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$ ( $n \neq -1$ )	$\frac{d}{dx} \left[ \frac{x^{n+1}}{n+1} + c \right]$ $= \cancel{(n+1)} \frac{x^{n+1-1}}{\cancel{(n+1)}} + 0$ $= x^n$
2.	$\int \frac{1}{x} dx = \log x + c$	$\frac{d}{dx} (\log x + c) = \frac{1}{x} + 0 = \frac{1}{x}$
3.	$\int e^x dx = e^x + c$	$\frac{d}{dx} (e^x) = e^x$
4.	$\int \sin x dx = -\cos x + c$	$\frac{d}{dx} (-\cos x + c) = \sin x$
5.	$\int \cos x dx = \sin x + c$	$\frac{d}{dx} (\sin x + c) = \cos x$
6.	$\int \sec^2 x dx = \tan x + c$	$\frac{d}{dx} (\tan x + c) = \sec^2 x$
7.	$\int \operatorname{cosec}^2 x dx = -\cot x + c$	$\frac{d}{dx} (-\cot x + c) = \operatorname{cosec}^2 x$
8.	$\int \sec x \tan x dx = \sec x + c$	$\frac{d}{dx} (\sec x + c) = \sec x \tan x$
9.	$\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$	$\frac{d}{dx} (-\operatorname{cosec} x + c) = \operatorname{cosec} x \cot x$

Particular forms of  $\int x^n dx = \frac{x^{n+1}}{n+1} + c$  where  $n \neq -1$ .

$$1. \int x dx = \frac{x^{1+1}}{1+1} = \frac{x^2}{2} + c$$

$$2. \int x^3 dx = \frac{x^{3+1}}{3+1} = \frac{x^4}{4} + c$$

$$3. \int dx = x + c \quad \because \frac{d}{dx} x = 1$$

$$4. \int \sqrt{x} \, dx = \int x^{1/2} \, dx = \frac{x^{1/2+1}}{\frac{1}{2}+1} = \frac{x^{3/2}}{\frac{3}{2}} = \frac{2}{3}x^{3/2} + c$$

$$5. \int \frac{1}{x^4} \, dx = \int x^{-4} \, dx = \frac{x^{-4+1}}{-4+1} = \frac{x^{-3}}{-3} + c$$

$$6. \int \frac{1}{x} \, dx = \log x + c \quad \because \frac{d}{dx}(\log x) = \frac{1}{x}$$

**Note:**

$$\int \frac{1}{x} \, dx = \int x^{-1} \, dx = \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} = \infty$$

So we should not use the formula  $\frac{x^{n+1}}{n+1}$

### Two Basic Theorems on Integration (Without Proof):

1. If  $u, v, w$  etc. are functions of  $x$ , then  $\int (u \pm v \pm w \dots) \, dx = \int u \, dx \pm \int v \, dx \pm \int w \, dx \pm \dots$

2. If  $f(x)$  is any function of  $x$  and  $K$  any constant then  $\int K f(x) \, dx = K \int f(x) \, dx$

**Example:**

1. Evaluate:  $\int (4x - 3x^2) \, dx$

**Solution:**

$$\begin{aligned} & \int (4x - 3x^2) \, dx \\ &= \int 4x \, dx - \int 3x^2 \, dx \\ &= 4 \int x \, dx - 3 \int x^2 \, dx \\ &= 4 \frac{x^{1+1}}{1+1} - 3 \frac{x^{2+1}}{2+1} + c \\ &= \frac{4x^2}{2} - \frac{3x^3}{3} + c \\ &= 2x^2 - x^3 + c \end{aligned}$$

### Integration using decomposition method:

In integration, there is no rule for multiplication (or) division of algebraic or trigonometric function as we have in differentiation. Such functions are to be decomposed into addition and subtraction before applying integration.

For example  $\frac{\sin^2 x}{1 + \cos x}$  can be decomposed as follows

$$\frac{\sin^2 x}{1 + \cos x} = \frac{1 - \cos^2 x}{1 + \cos x} = \frac{(1 + \cancel{\cos x})(1 - \cos x)}{(1 + \cancel{\cos x})} = 1 - \cos x,$$

which can be integrated using above theorems.

**Examples:**

1. Evaluate:  $\int (x+1)(x+2) dx$

**Solution:**

There is no  $uv$  rule in integration. So, we first multiply  $(x+1)$  and  $(x+2)$  and then integrate

$$\begin{aligned} & \int (x+1)(x+2) dx \\ &= \int (x^2 + 2x + x + 2) dx \\ &= \int (x^2 + 3x + 2) dx \\ &= \frac{x^3}{3} + \frac{3x^2}{2} + 2x + c \end{aligned}$$

2. Evaluate:  $\int (1+x^2)^3 dx$

**Solution:**

$$\begin{aligned} \text{Formula: } (a+b)^3 &= a^3 + b^3 + 3a^2b + 3ab^2 \\ (1+x^2)^3 &= 1^3 + (x^2)^3 + 3(1)x^2 + 3(1)(x^2)^2 \\ &= 1 + x^6 + 3x^2 + 3x^4 \end{aligned}$$

$$\begin{aligned} \int (1+x^2)^3 dx &= \int (1+x^6 + 3x^2 + 3x^4) dx \\ &= x + \frac{x^7}{7} + \frac{3x^3}{3} + \frac{3x^5}{5} + c \end{aligned}$$

3. Evaluate:  $\int \frac{\sin x}{1+\sin x} dx$

**Solution:**

$$\begin{aligned} & \int \frac{\sin x}{1+\sin x} dx \\ &= \int \frac{\sin x}{1+\sin x} \times \frac{(1-\sin x)}{(1-\sin x)} dx \end{aligned}$$

Multiply and divide by conjugate of  
 $1 + \sin x = 1 - \sin x$

$$\begin{aligned} & (a+b)(a-b) = a^2 - b^2 \\ &= \int \frac{\sin x(1-\sin x)}{1^2 - \sin^2 x} dx \end{aligned}$$

$$= \int \frac{\sin x - \sin^2 x}{\cos^2 x} dx$$

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \therefore 1 - \sin^2 x &= \cos^2 x \end{aligned}$$

Dividing separately

$$= \int \left( \frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \right) dx$$

$$= \int \frac{1}{\cos x} \frac{\sin x}{\cos x} dx - \int \tan^2 x dx$$

$$\begin{aligned} 1 + \tan^2 x &= \sec^2 x \\ \Rightarrow \tan^2 x &= \sec^2 x - 1 \end{aligned}$$

$$= \int \sec x \tan x dx - \int (\sec^2 x - 1) dx$$

$$= \sec x - (\tan x - x) \quad \text{Note: } \frac{d}{dx}(x) = 1 \Rightarrow \int 1 dx = x$$

$$= \sec x - \tan x + x + c$$

## 3.1 WORKED EXAMPLES

## PART – A

1. Evaluate :  $\int \left( x^3 + e^x + \frac{1}{x} \right) dx$

**Solution:**

$$\begin{aligned} & \int \left( x^3 + e^x + \frac{1}{x} \right) dx \\ &= \frac{x^4}{4} + e^x + \log x + c \end{aligned}$$

2. Evaluate:  $\int (\tan^2 x + \cot^2 x) dx$

**Solution:**

$$\begin{aligned} 1 + \tan^2 x &= \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1 \\ 1 + \cot^2 x &= \operatorname{cosec}^2 x \Rightarrow \cot^2 x = \operatorname{cosec}^2 x - 1 \\ \int (\tan^2 x + \cot^2 x) dx & \\ &= \int (\sec^2 x - 1 + \operatorname{cosec}^2 x - 1) dx \\ &= \int (\sec^2 x + \operatorname{cosec}^2 x - 2) dx \\ &= \tan x - \cot x - 2x + c \end{aligned}$$

3. Evaluate:  $\int (e^{3x} + e^{-5x}) dx$

**Solution:**

$$\begin{aligned} & \int (e^{3x} + e^{-5x}) dx \\ &= \frac{e^{3x}}{3} + \frac{e^{-5x}}{-5} \\ &= \frac{e^{3x}}{3} - \frac{e^{-5x}}{5} + c \end{aligned}$$

Hint : Divide by coefficient of x

In general  $\int e^{mx} dx = \frac{e^{mx}}{m}$  because

$$\frac{d}{dx} \left( \frac{e^{mx}}{m} \right) = \frac{1}{m} \frac{d}{dx} (e^{mx}) = \frac{1}{m} \cdot e^{mx} \cdot m = e^{mx}$$

4. Evaluate:  $\int \sec^2(3+4x) dx$

**Solution:**

$$\begin{aligned} & \int \sec^2(3+4x) dx \\ &= \frac{1}{4} \tan(3+4x) + c \end{aligned}$$

Hint : Divide by coefficient of x

Shown as  $\frac{1}{4}$

$$\frac{d}{dx} \left[ \frac{1}{4} \tan(3+4x) \right]$$

$$= \frac{1}{4} \sec^2 x (3+4x) 4 = \sec^2(3+4x)$$

5. Evaluate:  $\int (3-2x)^4 dx$

**Solution:**

$$\int (3-2x)^4 dx$$

Divide by coefficient of x

$$= \frac{1}{-2} \left[ \frac{(3-2x)^5}{5} \right] + c \quad \text{shown as } \frac{1}{-2}$$

6. Evaluate:  $\int \sqrt{1 - \sin 2x} \, dx$

**Solution:**

$$\int \sqrt{1 - \sin 2x} \, dx \quad \begin{array}{l} \sin^2 x + \cos^2 x = 1 \\ \sin 2x = 2 \sin x \cos x \end{array}$$

$$= \int \sqrt{\sin^2 x + \cos^2 x - 2 \sin x \cos x} \, dx$$

$$\boxed{a^2 + b^2 - 2ab = (a - b)^2}$$

$$= \int \sqrt{(\sin x - \cos x)^2} \, dx$$

$$= \int (\sin x - \cos x) \, dx$$

$$= -\cos x - \sin x + c$$

7. Evaluate:  $\int \sqrt{2x - 3} \, dx$

**Solution:**

$$\int \sqrt{2x - 3} \, dx$$

$$\int (2x - 3)^{1/2} \, dx \quad \boxed{\text{Divide by coefficient of } x}$$

$$= \frac{1}{2} \left[ \frac{(2x - 3)^{\frac{1}{2} + 1}}{\frac{1}{2} + 1} \right] + c$$

$$= \frac{1}{2} \left[ \frac{(2x - 3)^{3/2}}{3/2} \right] + c = \frac{1}{2} \times \frac{2}{3} (2x - 3)^{3/2} + c$$

$$= \frac{1}{3} (2x - 3)^{3/2} + c$$

8. Evaluate:  $\int \frac{dx}{\sqrt{2+x}}$

**Solution:**

$$\int \frac{dx}{\sqrt{2+x}} = \int \frac{dx}{(2+x)^{1/2}} = \int (2+x)^{-1/2} \, dx$$

$$= \frac{1}{1} \left[ \frac{(2+x)^{-1/2+1}}{-1/2+1} \right] + c \quad \boxed{\text{Divide by coefficient of } x = 1} \quad \text{shown as } \frac{1}{1}$$

$$= \frac{(2+x)^{1/2}}{\frac{1}{2}} + c = 2(2+x)^{1/2} + c$$

## PART – B &amp; C

1. Evaluate:  $\int (x+1)(2x-3) dx$

**Solution:**

There is no uv rule in integration.

$$\begin{aligned} & \int (x+1)(2x-3) dx \\ &= \int (2x^2 - 3x + 2x - 3) dx \\ &= \int (2x^2 - x - 3) dx \\ &= \frac{2x^3}{3} - \frac{x^2}{2} - 3x + C \end{aligned}$$

2. Evaluate:  $\int \left(x + \frac{1}{x}\right) \left(x^2 - \frac{1}{x^2}\right) dx$

**Solution:**

$$\begin{aligned} & \int \left(x + \frac{1}{x}\right) \left(x^2 - \frac{1}{x^2}\right) dx \\ &= \int \left(x^3 - \frac{1}{x} + x - \frac{1}{x^3}\right) dx \\ &= \int \left(x^3 - \frac{1}{x} + x - x^{-3}\right) dx \\ &= \frac{x^4}{4} - \log x + \frac{x^2}{2} - \frac{x^{-3+1}}{-3+1} + c \\ &= \frac{x^4}{4} - \log x + \frac{x^2}{2} - \frac{x^{-2}}{-2} + c \\ &= \frac{x^4}{4} - \log x + \frac{x^2}{2} + \frac{1}{2x^2} + c \end{aligned}$$

Do not write  $\frac{1}{x}$  as  $x^{-1}$

3. Evaluate:  $\int \left(\frac{2}{x^2} - \frac{7}{x} + \frac{3}{\sin^2 x}\right) dx$

**Solution:**

$$\begin{aligned} & \int \left(\frac{2}{x^2} - \frac{7}{x} + \frac{3}{\sin^2 x}\right) dx \\ &= \int \left(2x^{-2} - \frac{7}{x} + 3\operatorname{cosec}^2 x\right) dx \\ &= \frac{2x^{-2+1}}{-2+1} - 7\log x - 3\cot x + c \\ &= \frac{2x^{-1}}{-1} - 7\log x - 3\cot x + c \\ &= -\frac{1}{2x} - 7\log x - 3\cot x + c \end{aligned}$$

**Conjugate Model:**

4. Evaluate:  $\int \frac{\cos^2 x}{1 + \sin x} dx$

**Solution:**

Multiply and Divide by the conjugate of  
 $(1 + \sin x) = (1 - \sin x)$

$$\begin{aligned} & \int \frac{\cos^2 x}{1 + \sin x} \times \frac{(1 - \sin x)}{(1 - \sin x)} dx \\ & (a + b)(a - b) = a^2 - b^2 \\ & = \int \frac{\cos^2 x(1 - \sin x)}{1 - \sin^2 x} dx \\ & = \int \frac{\cos^2 x(1 - \sin x)}{\cos^2 x} dx \\ & = \int (1 - \sin x) dx \\ & = x + \cos x + c \end{aligned}$$

$$\begin{aligned} \cos^2 x + \sin^2 x &= 1 \\ \Rightarrow \cos^2 x &= 1 - \sin^2 x \end{aligned}$$

5. Evaluate:  $\int \sin^2 x dx$

**Solution:**

$$\begin{aligned} & \int \sin^2 x dx \\ & = \frac{1}{2} \int (1 - \cos 2x) dx \\ & = \frac{1}{2} \left[ x - \frac{\sin 2x}{2} \right] + c \end{aligned}$$

$$\begin{aligned} \cos 2x &= 1 - 2\sin^2 x \\ \Rightarrow 2\sin^2 x &= 1 - \cos 2x \\ \sin^2 x &= \frac{1}{2}(1 - \cos 2x) \end{aligned}$$

6. Evaluate:  $\int \cos^2 x dx$

**Solution:**

$$\begin{aligned} & \int \cos^2 x dx \\ & = \frac{1}{2} \int (1 + \cos 2x) dx \\ & = \frac{1}{2} \left[ x + \frac{\sin 2x}{2} \right] + c \end{aligned}$$

$$\begin{aligned} \cos 2x &= 2\cos^2 x - 1 \\ \Rightarrow 1 + \cos 2x &= 2\cos^2 x \\ \Rightarrow \frac{1}{2}(1 + \cos 2x) &= \cos^2 x \end{aligned}$$

7. Evaluate:  $\int \sin^3 x dx$

**Solution:**

$$\begin{aligned} & \int \sin^3 x dx \\ & = \frac{1}{4} \int (3\sin x - \sin 3x) dx \\ & = \frac{1}{4} \left[ -3\cos x + \frac{\cos 3x}{3} \right] + c \end{aligned}$$

$$\begin{aligned} \sin 3x &= 3\sin x - 4\sin^3 x \\ 4\sin^3 x &= 3\sin x - \sin 3x \\ \sin^3 x &= \frac{1}{4}(3\sin x - \sin 3x) \end{aligned}$$

8. Evaluate:  $\int \cos^3 x \, dx$ **Solution:**

$$\begin{aligned} & \int \cos^3 x \, dx \\ &= \frac{1}{4} \int (3 \cos x + \cos 3x) \, dx \\ &= \frac{1}{4} \left[ 3 \sin x + \frac{\sin 3x}{3} \right] + c \end{aligned}$$

$\begin{aligned} \cos 3x &= 4 \cos^3 x - 3 \cos x \\ 3 \cos x + \cos 3x &= 4 \cos^3 x \\ \frac{1}{4} (3 \cos x + \cos 3x) &= \cos^3 x \end{aligned}$
--

9. Evaluate:  $\int \sin 5x \cos 3x \, dx$ **Solution:**

$$\begin{aligned} & \therefore \int \sin 5x \cos 3x \, dx \\ &= \frac{1}{2} \int [\sin(5x + 3x) + \sin(5x - 3x)] \, dx \\ &= \frac{1}{2} \int (\sin 8x + \sin 2x) \, dx \\ &= \frac{1}{2} \left[ \frac{-\cos 8x}{8} - \frac{\cos 2x}{2} \right] + c \end{aligned}$$

$\begin{aligned} 2SC &= S + S \\ SC &= \frac{1}{2}[S + S] \end{aligned}$	<p>Recall from Maths-I</p> $\begin{aligned} S + S &= 2SC \\ S - S &= 2CS \\ C + C &= 2CC \\ C - C &= -2SS \end{aligned}$
--	--

10. Evaluate:  $\int \sin 7x \sin 4x \, dx$ **Solution:**

$$\begin{aligned} & \int \sin 7x \sin 4x \, dx \\ &= \frac{-1}{2} \int [\cos(7x + 4x) - \cos(7x - 4x)] \, dx \\ &= -\frac{1}{2} \int (\cos 11x - \cos 3x) \, dx \\ &= -\frac{1}{2} \left[ \frac{\sin 11x}{11} - \frac{\sin 3x}{3} \right] + c \end{aligned}$$

$\begin{aligned} -2SS &= C - C \\ SS &= -\frac{1}{2}[C - C] \end{aligned}$
--

11. Evaluate:  $\int \sin^2 3x \, dx$ **Solution:**

$$\begin{aligned} & \int \sin^2 3x \, dx \\ &= \frac{1}{2} \int (1 - \cos 6x) \, dx \\ &= \frac{1}{2} \left[ x - \frac{\sin 6x}{6} \right] + c \end{aligned}$$

<p>Refer Problem 5</p> $\begin{aligned} \sin^2 x &= \frac{1}{2}[1 - \cos 2x] \\ \sin^2 3x &= \frac{1}{2}[1 - \cos 2(3x)] \\ &= \frac{1}{2}[1 - \cos 6x] \end{aligned}$
--

12. Evaluate:  $\int \cos^3 5x \, dx$ **Solution:**

$$\begin{aligned} & \int \cos^3 5x \, dx \\ &= \frac{1}{4} \int (3 \cos 5x + \cos 15x) \, dx \\ &= \frac{1}{4} \left[ \frac{3 \sin 5x}{5} + \frac{\sin 15x}{15} \right] + c \end{aligned}$$

<p>Refer Problem 8</p> $\begin{aligned} \cos^3 x &= \frac{1}{4}(3 \cos x + \cos 3(5x)) \\ \cos^3 5x &= \frac{1}{4}(3 \cos 5x + \cos 3x) \\ &= \frac{1}{4}(3 \cos 5x + \cos 15x) \end{aligned}$
--

## 4.2 INTEGRATION BY SUBSTITUTION

So far we have dealt with functions, either directly integrable using integration formula (or) integrable after decomposing the given functions into sums & differences.

But there are functions like  $\frac{\sin(\log x)}{x}$ ,  $\frac{2x+3}{x^2+3x+5}$  which cannot be decomposed into sums (or) differences of simple functions.

In these cases, using proper substitution, we shall reduce the given form into standard form, which can be integrated using basic integration formula.

When the integrand (the function to be integrated) is either in multiplication or in division form and if the derivative of one full or meaningful part of the function is equal to the other function then the integration can be evaluated using substitution method as given in the following examples.

$$1. \int \frac{2x+3}{x^2+3x+5} \text{ since } \frac{d}{dx} (x^2+3x+5) \text{ is } 2x+3 \text{ it can be integrated by taking } y = x^2+3x+5.$$

$$2. \int \frac{\sin(\log x)}{x} dx = \int \sin(\log x) \frac{1}{x} dx$$

$$\text{Here } \frac{d}{dx} (\log x) = \frac{1}{x}$$

The above integration can be evaluated by taking  $y = \log x$ .

### Integrals of some standard forms:

Integrals of the form  $\int [f(x)]^n f'(x) dx$ ,  $\int \frac{f'(x)}{f(x)} dx$ ,  $\int F(f(x)) f'(x) dx$  are all, more or less of the same type and the use of substitution  $y = f(x)$  will reduce the given function to simple standard form which can be integrated using integration formulae.

## 4.2 WORKED EXAMPLES

### PART – A

$$1. \text{ Evaluate: } I = \int \sin^3 x \boxed{\cos x dx}.$$

**Solution:**

$$\text{Put } y = \sin x \quad \dots\dots\dots(1)$$

$$\frac{dy}{dx} = \cos x$$

$$dy = \boxed{\cos x dx}$$

$$I = \int y^3 dy \text{ using (1)}$$

$$= \frac{y^4}{4} + c$$

$$= \frac{\sin^4 x}{4} + c$$

2. Evaluate:  $I = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

**Solution:**

$$y = e^x + e^{-x} \quad \dots\dots\dots(1)$$

$$\frac{dy}{dx} = e^x + e^{-x} (-1)$$

$$= e^x - e^{-x}$$

$$dy = (e^x - e^{-x}) dx$$

$$\therefore I = \int \frac{dy}{y} \text{ using (1)}$$

$$= \log y + c$$

$$= \log (e^x + e^{-x}) + c$$

3. Evaluate:  $\int \tan x dx$

**Solution:**

$$I = \int \frac{\sin x}{\cos x} dx$$

Put  $y = \cos x \quad \dots\dots\dots(1)$

$$\frac{dy}{dx} = -\sin x$$

$$dy = -\sin x dx$$

$$-dy = \sin x dx$$

$$I = \int \frac{-dy}{y} = -\log y = -\log (\cos x) + c$$

$$= \log (\cos x)^{-1} = \log \left( \frac{1}{\cos x} \right) = \log \sec x + c$$

Note :

$$\frac{d}{dx} \log(\sec x)$$

$$= \frac{1}{\sec x} \cdot \sec x \tan x$$

$$= \tan x$$

$$\therefore \int \tan x dx = \log(\sec x) + c$$

4. Evaluate:  $\int \cot x dx$

**Solution:**

$$I = \int \cot x dx$$

$$I = \int \frac{\cos x}{\sin x} dx$$

Put  $y = \sin x \quad \dots\dots\dots(1)$

$$\frac{dy}{dx} = \cos x$$

$$dy = \cos x dx$$

$$\therefore I = \int \frac{dy}{y} = \log y = \log(\sin x) + c$$

Note :

$$\frac{d}{dx} \log(\sin x)$$

$$= \frac{1}{\sin x} \cdot \cos x$$

$$= \cot x$$

$$\therefore \int \cot x dx = \log(\sin x) + c$$

5. Evaluate:  $\int e^{\sin^{-1}x} \frac{1}{\sqrt{1-x^2}} dx$

**Solution:**

$$I = \int e^{\sin^{-1}x} \frac{1}{\sqrt{1-x^2}} dx$$

Put  $y = \sin^{-1}x$  .....(1)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$dy = \frac{1}{\sqrt{1-x^2}} dx$$

$$\begin{aligned} \therefore I &= \int e^y dy = e^y + c \\ &= e^{\sin^{-1}x} + c \end{aligned}$$

6. Evaluate:  $\int \frac{(\log x)^5}{x} dx$

**Solution:**

$$I = \int (\log x)^5 \frac{1}{x} dx$$

Put  $y = \log x$  .....(1)

$$\frac{dy}{dx} = \frac{1}{x}$$

$$dy = \frac{1}{x} dx$$

$$I = \int y^5 dy$$

$$= \frac{y^6}{6} + c$$

$$= \frac{(\log x)^6}{6} + c$$

7. Evaluate:  $\int \frac{\cos x}{2+3\sin x} dx$

**Solution:**

$$I = \int \frac{\cos x}{2+3\sin x} dx$$

$$y = 2 + 3\sin x$$

$$\frac{dy}{dx} = 3\cos x$$

$$dy = 3\cos x dx$$

$$\frac{1}{3} dy = \cos x dx$$

$$\begin{aligned}\therefore I &= \int \frac{1}{3} \frac{dy}{y} \\ &= \frac{1}{3} \log y + c \\ &= \frac{1}{3} \log(2 + 3 \sin x) + c\end{aligned}$$

8. Evaluate:  $\int \sec x \, dx$

**Solution:**

$$\begin{aligned}I &= \int \frac{\sec x (\sec x + \tan x) \, dx}{(\sec x + \tan x)} \\ &= \int \frac{(\sec^2 x + \sec x \tan x)}{(\sec x + \tan x)} \, dx\end{aligned}$$

Multiply and Divide by
$\sec x + \tan x$

$$y = \sec x + \tan x$$

$$\frac{dy}{dx} = \sec x \tan x + \sec^2 x$$

$$dy = (\sec x \tan x + \sec^2 x) \, dx$$

$$\begin{aligned}\therefore I &= \int \frac{dy}{y} \\ &= \log y + c \\ &= \log(\sec x + \tan x) + c\end{aligned}$$

9. Evaluate:  $\int \operatorname{cosec} x \, dx$

**Solution:**

$$\begin{aligned}I &= \int \operatorname{cosec} x \, dx \\ &= \int \frac{\operatorname{cosec} x (\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x + \cot x)} \, dx \\ &= \int \frac{(\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x)}{(\operatorname{cosec} x + \cot x)} \, dx\end{aligned}$$

$$y = \operatorname{cosec} x + \cot x$$

$$\frac{dy}{dx} = -\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x$$

$$dy = -(\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x) \, dx$$

$$-dy = (\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x) \, dx$$

$$\begin{aligned}\therefore I &= \int -\frac{dy}{y} \\ &= -\log(\operatorname{cosec} x + \cot x) + c\end{aligned}$$

10. Evaluate:  $\int (x^2 - 5)^4 x \, dx$

**Solution:**

$$I = \int (x^2 - 5)^4 \boxed{x \, dx}$$

$$y = x^2 - 5$$

$$\frac{dy}{dx} = 2x$$

$$dy = 2x \, dx$$

$$\frac{1}{2} dy = \boxed{x \, dx}$$

$$\therefore I = \int y^4 \frac{1}{2} dy$$

$$= \frac{1}{2} \frac{y^5}{5} + c$$

$$= \frac{1}{10} y^5 + c$$

$$= \frac{1}{10} (x^2 - 5)^5 + c$$

## PART B & C

1. Evaluate:  $\int (2 + \sin x)^3 \boxed{\cos x \, dx}$

**Solution:**

$$y = 2 + \sin x$$

$$\frac{dy}{dx} = \cos x$$

$$dy = \boxed{\cos x \, dx}$$

$$\therefore I = \int y^3 dy$$

$$= \frac{y^4}{4} + c = \frac{(2 + \sin x)^4}{4} + c$$

2. Evaluate:  $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

**Solution:**

$$I = \int \cos \sqrt{x} \boxed{\frac{1}{\sqrt{x}} dx}$$

$$y = \sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$2dy = \boxed{\frac{1}{\sqrt{x}} dx}$$

$$\begin{aligned}\therefore I &= \int \cos y 2 dy \\ &= 2 \sin y + c \\ &= 2 \sin(\sqrt{x}) + c\end{aligned}$$

3. Evaluate:  $\int \frac{\sec^2 x}{(2 + 3 \tan x)^3} dx$

**Solution:**

$$I = \int \frac{\boxed{\sec^2 x \, dx}}{(2 + 3 \tan x)^3}$$

$$y = 2 + 3 \tan x$$

$$\frac{dy}{dx} = 3 \sec^2 x$$

$$dy = 3 \sec^2 x \, dx$$

$$\frac{1}{3} dy = \boxed{\sec^2 x \, dx}$$

$$\begin{aligned}\therefore I &= \frac{1}{3} \int \frac{dy}{y^3} \\ &= \frac{1}{3} \int y^{-3} dy \\ &= \frac{1}{3} \frac{y^{-3+1}}{-3+1} + c \\ &= \frac{1}{3} \cdot \frac{y^{-2}}{-2} = \frac{1}{-6} y^{-2} + c \\ &= -\frac{1}{6} (2 + 3 \tan x)^{-2} + c\end{aligned}$$

4. Evaluate:  $\int x^2 \cos(x^3) \, dx$

**Solution:**

$$I = \int x^2 \cos(x^3) \, dx = \int \cos(x^3) \boxed{x^2 dx}$$

$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

$$dy = 3x^2 dx$$

$$\frac{1}{3} dy = \boxed{x^2 dx}$$

$$\begin{aligned}\therefore I &= \int \cos y \frac{1}{3} dy \\ &= \frac{1}{3} \sin y \\ &= \frac{1}{3} \sin(x^3) + c\end{aligned}$$

5. Evaluate:  $\int e^{\sin^2 x} \sin 2x \, dx$

**Solution:**

$$I = \int e^{\sin^2 x} \sin 2x \, dx \quad \boxed{\therefore \sin 2A = 2 \sin A \cos A}$$

$$= \int e^{\sin^2 x} \boxed{2 \sin x \cos x \, dx}$$

$$y = \sin^2 x$$

$$\frac{dy}{dx} = 2 \sin x \cos x$$

$$dy = \boxed{2 \sin x \cos x \, dx}$$

$$\therefore I = \int e^y dy$$

$$= e^y + c$$

$$= e^{\sin^2 x} + c$$

6. Evaluate:  $\int (2x^2 - 8x + 5)^{11} (x - 2) \, dx$

**Solution:**

$$I = \int (2x^2 - 8x + 5)^{11} \boxed{(x - 2) \, dx}$$

$$y = 2x^2 - 8x + 5$$

$$\frac{dy}{dx} = 4x - 8$$

$$\frac{dy}{dx} = 4(x - 2)$$

$$dy = 4(x - 2) dx$$

$$\frac{1}{4} dy = \boxed{(x - 2) \, dx}$$

$$\therefore I = \int y^{11} \frac{1}{4} dy$$

$$= \frac{1}{4} \frac{y^{12}}{12} + C$$

$$= \frac{1}{48} y^{12} + C$$

$$= \frac{1}{48} (2x^2 - 8x + 5)^{12} + C$$

7. Evaluate:  $\int \frac{(\sin^{-1} x)^4 dx}{\sqrt{1-x^2}}$

**Solution:**

$$I = \int (\sin^{-1} x)^4 \boxed{\frac{1}{\sqrt{1-x^2}} dx}$$

$$y = \sin^{-1} x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$dy = \boxed{\frac{1}{\sqrt{1-x^2}} dx}$$

$$\begin{aligned}
 I &= \int y^4 dy \\
 &= \frac{y^5}{5} + C \\
 &= \frac{(\sin^{-1} x)^5}{5} + C
 \end{aligned}$$

8. Evaluate:  $\int \sqrt{\tan x} \sec^2 x \, dx$

**Solution:**

$$\begin{aligned}
 &\int \sqrt{\tan x} \boxed{\sec^2 x \, dx} \\
 y &= \tan x \\
 \frac{dy}{dx} &= \sec^2 x \\
 dy &= \boxed{\sec^2 x \, dx} \\
 \therefore I &= \int \sqrt{y} dy \\
 &= \int y^{1/2} dy \\
 &= \frac{y^{1/2+1}}{\frac{1}{2}+1} = \frac{y^{3/2}}{\frac{3}{2}} = \frac{2}{3} y^{3/2} \\
 &= \frac{2}{3} (\tan x)^{3/2} + C
 \end{aligned}$$

9. Evaluate:  $\int \frac{1}{x \log x} dx$

**Solution:**

$$\begin{aligned}
 I &= \int \frac{1}{\log x} \boxed{\frac{1}{x} dx} \\
 y &= \log x \\
 \frac{dy}{dx} &= \frac{1}{x} \\
 dy &= \boxed{\frac{1}{x} dx} \\
 I &= \int \frac{1}{y} dy \\
 &= \log y + C \\
 &= \log (\log x) + C
 \end{aligned}$$

10. Evaluate:  $\int \frac{\cot x}{\log(\sin x)} dx$

**Solution:**

$$I = \int \frac{1}{\log(\sin x)} \cdot \boxed{\cot x \, dx}$$

$$y = \log(\sin x)$$

$$\frac{dy}{dx} = \frac{1}{\sin x} \cos x$$

$$\frac{dy}{dx} = \cot x$$

$$dy = \boxed{\cot x \, dx}$$

$$\therefore I = \int \frac{1}{y} dy$$

$$= \log y + C$$

$$= \log(\log(\sin x)) + C$$

### 4.3 STANDARD INTEGRALS

Integrals of the form  $\int \frac{dx}{a^2 \pm x^2}$ ,  $\int \frac{dx}{x^2 - a^2}$  and  $\int \frac{dx}{\sqrt{a^2 - x^2}}$ .

1. Evaluate:  $\int \frac{1}{a^2 + x^2} dx$

$$I = \int \frac{1}{a^2 + x^2} \boxed{dx}$$

Put  $x = a \tan \theta$

$$\frac{dx}{d\theta} = a \sec^2 \theta$$

$$dx = \boxed{a \sec^2 \theta d\theta}$$

$$\begin{aligned} & a^2 + x^2 \\ &= a^2 + (a \tan \theta)^2 \\ &= a^2 + a^2 \tan^2 \theta \\ &= a^2(1 + \tan^2 \theta) \\ &= a^2 \sec^2 \theta \end{aligned}$$

$$\therefore I = \int \frac{1}{a^2 \sec^2 \theta} \cdot a \sec^2 \theta d\theta$$

$$= \int \frac{1}{a} d\theta$$

$$= \frac{1}{a} \theta$$

$$= \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C$$

$$\begin{aligned} x &= a \tan \theta \\ \Rightarrow \tan \theta &= \frac{x}{a} \\ \Rightarrow \theta &= \tan^{-1} \left( \frac{x}{a} \right) \end{aligned}$$

2. Evaluate:  $\int \frac{dx}{a^2 - x^2}$

$$I = \int \frac{1 \cdot dx}{(a+x)(a-x)}$$

$$\begin{aligned} (a+x) + (a-x) &= 2a \\ \therefore \frac{1}{2a} [(a+x) + (a-x)] &= 1 \end{aligned}$$

$$= \int \frac{\frac{1}{2a} [(a+x) + (a-x)]}{(a+x)(a-x)} dx$$

$$= \frac{1}{2a} \int \frac{a+x+a-x}{(a+x)(a-x)} dx$$

Dividing Separately

$$= \frac{1}{2a} \int \frac{\cancel{(a+x)}}{\cancel{(a+x)}(a-x)} dx + \frac{1}{2a} \int \frac{\cancel{(a-x)}}{(a+x)\cancel{(a-x)}} dx$$

$$= \frac{1}{2a} \int \frac{dx}{a-x} + \frac{1}{2a} \int \frac{dx}{a+x}$$

$$\text{Note: } \frac{d}{dx}(a-x) = -1 \text{ and } \frac{d}{dx}(x+a) = 1$$

$$I = \frac{-1}{2a} \int \frac{(-1) dx}{(a-x)} + \frac{1}{2a} \int \frac{1 dx}{a+x}$$

$$= \frac{-1}{2a} \log(a-x) + \frac{1}{2a} \log(a+x)$$

$$= \frac{1}{2a} \log(a+x) - \frac{1}{2a} \log(a-x)$$

$$= \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right) + c$$

3. Evaluate:  $\int \frac{dx}{x^2 - a^2}$

$$I = \int \frac{dx}{x^2 - a^2}$$

$$= \int \frac{dx}{(x+a)(x-a)}$$

$$\begin{aligned} (x+a) - (x-a) &= \\ = x+a-x+a &= \\ = 2a & \end{aligned}$$

$$= \int \frac{1}{2a} \frac{(x+a) - (x-a)}{(x+a)(x-a)} dx$$

$$= \frac{1}{2a} \int \frac{(x+a) - (x-a)}{(x+a)(x-a)} dx$$

Dividing Separately,

$$= \frac{1}{2a} \int \frac{\cancel{(x+a)} dx}{\cancel{(x+a)}(x-a)} - \frac{1}{2a} \int \frac{\cancel{(x-a)} dx}{(x+a)\cancel{(x-a)}}$$

$$= \frac{1}{2a} \int \frac{1}{x-a} dx - \frac{1}{2a} \int \frac{1}{x+a} dx$$

$$= \frac{1}{2a} \log(x-a) - \frac{1}{2a} \log(x+a)$$

$$= \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right) + c$$

4. Evaluate:  $\int \frac{1}{\sqrt{a^2 - x^2}} dx$

$$I = \int \frac{1}{\sqrt{a^2 - x^2}} dx$$

$$\begin{array}{l|l} \text{Put } x = a \sin \theta & x = a \sin \theta \\ a^2 - x^2 = a^2 - (a \sin \theta)^2 & \frac{dx}{d\theta} = a \cos \theta \\ = a^2 - a^2 \sin^2 \theta & dx = \boxed{a \cos \theta d\theta} \\ = a^2 (1 - \sin^2 \theta) & \\ = a^2 \cos^2 \theta & \end{array}$$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 \cos^2 \theta} = a \cos \theta$$

$$\begin{array}{l} \therefore I = \int \frac{1}{a \cos \theta} \cdot a \cos \theta d\theta \\ = \int d\theta \\ = \theta + c \\ = \sin^{-1} \frac{x}{a} + c \end{array} \quad \begin{array}{l} x = a \sin \theta \\ \Rightarrow \sin \theta = \frac{x}{a} \\ \Rightarrow \theta = \sin^{-1} \left( \frac{x}{a} \right) \end{array}$$

### List of Standard Integrals Formulae:

Sl.No.	Integration	Result
1.	$\int \frac{dx}{a^2 + x^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a} + c$
2.	$\int \frac{dx}{a^2 - x^2}$	$\frac{1}{2a} \log \frac{a+x}{a-x} + c$
3.	$\int \frac{dx}{x^2 - a^2}$	$\frac{1}{2a} \log \left( \frac{x-a}{x+a} \right) + c$
4.	$\int \frac{dx}{\sqrt{a^2 - x^2}}$	$\sin^{-1} \left( \frac{x}{a} \right) + c$

## 4.3 WORKED EXAMPLES

### PART - A

1. Evaluate:  $\int \frac{dx}{9 + x^2}$

**Solution:**

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + c$$

$$\boxed{a^2 = 9 \Rightarrow a = 3}$$

$$\therefore \int \frac{dx}{x^2 + 9} = \int \frac{dx}{x^2 + 3^2} = \frac{1}{3} \tan^{-1} \left( \frac{x}{3} \right) + c$$

2. Evaluate:  $\int \frac{dx}{25-x^2}$

**Solution:**

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right) + c$$

$$\boxed{\text{If } a=5, \text{ then } a^2=25}$$

$$\begin{aligned} \therefore \int \frac{dx}{25-x^2} &= \int \frac{dx}{5^2-x^2} \\ &= \frac{1}{2 \times 5} \log\left(\frac{5+x}{5-x}\right) + c \\ &= \frac{1}{10} \log\left(\frac{5+x}{5-x}\right) + c \end{aligned}$$

3. Evaluate:  $\int \frac{dx}{x^2-16}$

**Solution:**

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right)$$

$$\boxed{a^2=16 \Rightarrow a=4}$$

$$\begin{aligned} \therefore \int \frac{dx}{x^2-4^2} &= \frac{1}{2 \times 4} \log\left(\frac{x-4}{x+4}\right) + c \\ &= \frac{1}{8} \log\left(\frac{x-4}{x+4}\right) + c \end{aligned}$$

4. Evaluate:  $\int \frac{dx}{\sqrt{36-x^2}}$

**Solution:**

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\boxed{a^2=36 \Rightarrow a=6}$$

$$\therefore \int \frac{dx}{\sqrt{6^2-x^2}} = \sin^{-1}\left(\frac{x}{6}\right) + c$$

## PART – B & C

1. Evaluate:  $\int \frac{dx}{(2x+3)^2+9}$

**Solution:**

$$I = \int \frac{\boxed{dx}}{(2x+3)^2+9}$$

$$y = 2x + 3$$

$$\frac{dy}{dx} = 2$$

$$dy = 2dx$$

$$\frac{1}{2} dy = \boxed{dx}$$

$$\begin{aligned} \therefore I &= \frac{1}{2} \int \frac{dy}{y^2 + 9} && \boxed{a^2 = 9 \Rightarrow a = 3} \\ &= \frac{1}{2} \times \frac{1}{a} \tan^{-1} \left( \frac{y}{a} \right) + c \\ &= \frac{1}{2} \times \frac{1}{3} \tan^{-1} \left( \frac{y}{3} \right) + c \\ &= \frac{1}{6} \tan^{-1} \left( \frac{2x+3}{3} \right) + c \end{aligned}$$

2. Evaluate:  $\int \frac{\boxed{dx}}{(3x-4)^2 - 25}$

**Solution:**

Put  $y = 3x - 4$

$$\frac{dy}{dx} = 3$$

$$dy = 3dx$$

$$\frac{1}{3} dy = \boxed{dx}$$

$$\begin{aligned} &= \frac{1}{3} \int \frac{dy}{y^2 - 5^2} && a^2 = 25 \Rightarrow a = 5 \\ &= \frac{1}{3} \times \frac{1}{2a} \log \left[ \frac{y-5}{y+5} \right] \\ &= \frac{1}{3} \times \frac{1}{2 \times 5} \log \left[ \frac{(3x-4)-5}{3x-4+5} \right] + c && = \\ &= \frac{1}{30} \log \left[ \frac{3x-9}{3x+1} \right] + c \end{aligned}$$

3. Evaluate:  $\int \frac{dx}{49 - 4x^2}$

**Solution:**

$$I = \int \frac{\boxed{dx}}{49 - (2x)^2} && \boxed{(2x)^2 = 4x^2}$$

$$y = 2x$$

$$\frac{dy}{dx} = 2$$

$$dy = 2dx$$

$$\frac{1}{2} dy = \boxed{dx}$$

$$\begin{aligned} \therefore I &= \frac{1}{2} \int \frac{dy}{7^2 - y^2} \quad \boxed{a^2 = 49 \Rightarrow a = 7} \\ &= \frac{1}{2} \times \frac{1}{2a} \log \left[ \frac{7+y}{7-y} \right] \\ &= \frac{1}{2} \times \frac{1}{2 \times 7} \log \left[ \frac{7+2x}{7-2x} \right] + c \\ &= \frac{1}{28} \log \left[ \frac{7+2x}{7-2x} \right] + c \end{aligned}$$

4. Evaluate:  $\int \frac{dx}{\sqrt{121 - (3x+5)^2}}$

**Solution:**

$$\begin{aligned} I &= \int \frac{\boxed{dx}}{\sqrt{121 - (3x+5)^2}} \\ y &= 3x+5 \\ \frac{dy}{dx} &= 3 \\ dy &= 3dx \\ \frac{1}{3} dy &= \boxed{dx} \\ \therefore I &= \frac{1}{3} \int \frac{dy}{\sqrt{11^2 - y^2}} \quad \boxed{a^2 = 121 \Rightarrow a = 11} \\ &= \frac{1}{3} \times \sin^{-1} \left( \frac{y}{a} \right) \\ &= \frac{1}{3} \sin^{-1} \left( \frac{y}{11} \right) = \frac{1}{3} \sin^{-1} \left( \frac{3x+5}{11} \right) + c \end{aligned}$$

5. Evaluate:  $\int \frac{dx}{\sqrt{144 - 5x^2}}$

**Solution:**

$$\begin{aligned} I &= \int \frac{dx}{\sqrt{144 - 5x^2}} \\ &= \int \frac{\boxed{dx}}{\sqrt{12^2 - (\sqrt{5}x)^2}} \quad \boxed{(\sqrt{5}x)^2 = 5x^2} \\ \text{Put } y &= \sqrt{5}x \\ \frac{dy}{dx} &= \sqrt{5} \\ dy &= \sqrt{5}dx \\ \frac{1}{\sqrt{5}} dy &= \boxed{dx} \end{aligned}$$

$$\begin{aligned}
 I &= \frac{1}{\sqrt{5}} \int \frac{dy}{\sqrt{12^2 - y^2}} \\
 &= \frac{1}{\sqrt{5}} \sin^{-1} \left( \frac{y}{12} \right) + c \\
 &= \frac{1}{\sqrt{5}} \sin^{-1} \left( \frac{\sqrt{5}x}{12} \right) + c
 \end{aligned}$$

## EXERCISES

### PART – A

1. Evaluate the following:

$$\begin{array}{lll}
 \text{(i)} \int (2x^3 - 4\sqrt{x}) dx & \text{(ii)} \int \left( x^3 - \frac{7}{x^2} + \frac{1}{x} \right) dx & \text{(iii)} \int \left( \frac{7}{\cos^2 x} - \frac{3}{\sin^2 x} \right) dx \\
 \text{(iv)} \int \frac{(e^{4x} + e^{-4x})}{2} dx & \text{(v)} \int \sec 5x \tan 5x dx & \text{(vi)} \int e^{8x-5} dx
 \end{array}$$

2. Evaluate the following:

$$\begin{array}{lll}
 \text{(i)} \int (2 - 5x)^4 dx & \text{(ii)} \int \frac{dx}{\sqrt{4 - x^2}} & \text{(iii)} \int \frac{dx}{4 + x^2} \\
 \text{(iv)} \int \sqrt{\frac{1 + \cos 2x}{2}} dx & \text{(v)} \int \sqrt{3 + 4x} dx & \text{(vi)} \int \frac{dx}{\sqrt{1 - x}}
 \end{array}$$

3. Evaluate the following:

$$\begin{array}{lll}
 \text{(i)} \int \cos^5 x \sin x dx & \text{(ii)} \int \frac{e^{5x}}{1 + e^{5x}} dx & \text{(iii)} \int e^{\tan^{-1} x} \frac{1}{1 + x^2} dx \\
 \text{(iv)} \int \frac{\log x}{x} dx & \text{(v)} \int \frac{\sin x}{3 \cos x + 4} dx & \text{(vi)} \int (x^2 + 3)^4 x dx
 \end{array}$$

4. Evaluate the following:

$$\begin{array}{llll}
 \text{(i)} \int \frac{dx}{49 + x^2} & \text{(ii)} \int \frac{dx}{4 - x^2} & \text{(iii)} \int \frac{dx}{x^2 - 36} & \text{(iv)} \int \frac{dx}{\sqrt{64 - x^2}}
 \end{array}$$

### PART B and C

1. Evaluate the following:

$$\begin{array}{lll}
 \text{(i)} \int (3x - 4)(2x + 5) dx & \text{(ii)} \int \left( x - \frac{1}{x} \right) \left( 3 + \frac{5}{x^2} \right) dx & \text{(iii)} \int \frac{(x^5 - x^3 + x^2)}{x^3} dx \\
 \text{(iv)} \int \frac{\sin^2 x dx}{1 - \cos x} & \text{(v)} \int \frac{1}{1 + \cos x} dx & \text{(vi)} \int \frac{1}{1 - \sin x} dx
 \end{array}$$

2. Evaluate the following:

$$\begin{array}{lll}
 \text{(i)} \int \sin^2 4x dx & \text{(ii)} \int \cos^2 3x dx & \text{(iii)} \int \sin^3 6x dx \\
 \text{(iv)} \int \cos^3 2x dx & \text{(v)} \int \cos 11x \sin 7x dx & \text{(vi)} \int \cos 6x \cos 2x dx
 \end{array}$$

3. Evaluate the following:

(i)  $\int (3 - \cos x)^5 \sin x \, dx$

(ii)  $\int \frac{\sin \sqrt{x}}{\sqrt{x}} \, dx$

(iii)  $\int \frac{\operatorname{cosec}^2 x \, dx}{4 + 5 \cot x}$

(iv)  $\int x^2 \sec^2(x^3) \, dx$

(v)  $\int e^{\cos^2 x} \sin 2x \, dx$

(vi)  $\int (x^2 - 6x + 5)^7 (x - 3) \, dx$

4. Evaluate the following:

(i)  $\int \frac{(\tan^{-1} x)^3}{1 + x^2} \, dx$

(ii)  $\int \sqrt{\sin x} \cos x \, dx$

(iii)  $\int \frac{1}{x(\log x)^2} \, dx$

(iv)  $\int \frac{\tan x \, dx}{\log(\sec x)}$

(v)  $\int e^{2\sin^{-1} x} \frac{1}{\sqrt{1-x^2}} \, dx$

(vi)  $\int \frac{(2x+3)}{\sqrt{x^2+3x-4}} \, dx$

5. Evaluate the following:

(i)  $\int \frac{dx}{4 + (3x+1)^2}$

(ii)  $\int \frac{dx}{25 + 4x^2}$

(iii)  $\int \frac{dx}{4 - (7x-3)^2}$

(iv)  $\int \frac{dx}{(5x+2)^2 - 4}$

(v)  $\int \frac{dx}{\sqrt{25 - (x+1)^2}}$

(vi)  $\int \frac{dx}{\sqrt{169 - 4x^2}}$

## ANSWERS

### PART - A

1. (i)  $\frac{2x^4}{4} - \frac{8}{3}x^{3/2} + c$

(ii)  $\frac{x^4}{4} + \frac{7}{x} + \log x + c$

(iii)  $7 \tan x + 3 \cot x + c$

(iv)  $\frac{1}{2} \left[ \frac{e^{4x}}{4} + \frac{e^{-4x}}{-4} \right] + c$

(v)  $\frac{\sec 5x}{5} + c$

(vi)  $\frac{1}{8} e^{8x-5} + c$

2. (i)  $\frac{-1}{25} (2-5x)^5 + c$

(ii)  $\sin^{-1} \left( \frac{x}{2} \right) + c$

(iii)  $\frac{1}{2} \tan^{-1} \left( \frac{x}{2} \right) + c$

(iv)  $\sin x + c$

(v)  $\frac{1}{6} (3+4x)^{3/2} + c$

(vi)  $-2\sqrt{1-x} + c$

3. (i)  $-\frac{\cos^6 x}{6} + c$

(ii)  $\frac{1}{5} \log(1 + e^{5x}) + c$

(iii)  $e^{\tan^{-1} x} + c$

(iv)  $\frac{1}{2} (\log x)^2 + c$

(v)  $-\frac{1}{3} \log(3 \cos x + 4) + c$

(vi)  $\frac{1}{10} (x^2 + 3)^5 + c$

4. (i)  $\frac{1}{7} \tan^{-1} \left( \frac{x}{7} \right) + c$

(ii)  $\frac{1}{4} \log \left( \frac{2+x}{2-x} \right) + c$

(iii)  $\frac{1}{12} \log \left( \frac{x-6}{x+6} \right) + c$

(iv)  $\sin^{-1} \left( \frac{x}{8} \right) + c$

### PART - B & C

1. (i)  $2x^3 + \frac{7x^2}{2} - 20x + c$

(ii)  $\frac{3x^2}{2} + 2 \log x + \frac{5}{2} x^{-2} + c$

(iii)  $\frac{x^3}{3} - x + \log x + c$

(iv)  $x + \sin x + c$

(v)  $-\cot x + \operatorname{cosec} x + c$

(vi)  $\tan x + \sec x + c$

2. (i)  $\frac{1}{2}\left[x - \frac{\sin 8x}{8}\right] + c$       (ii)  $\frac{1}{2}\left[x + \frac{\sin 6x}{6}\right] + c$       (iii)  $\frac{1}{4}\left[\frac{-3\cos 6x}{6} + \frac{\cos 18x}{18}\right] + c$   
 (iv)  $\frac{1}{4}\left[\frac{3\sin 2x}{2} + \frac{\sin 6x}{6}\right] + c$       (v)  $\frac{1}{2}\left[\frac{-\cos 18x}{18} + \frac{\cos 4x}{4}\right] + c$       (vi)  $\frac{1}{2}\left[\frac{\sin 8x}{8} + \frac{\sin 4x}{4}\right] + c$
3. (i)  $-\frac{1}{6}(3 - \cos x)^6 + c$       (ii)  $-2\cos\sqrt{x} + c$       (iii)  $-\frac{1}{5}\log(4 + 5\cot x) + c$   
 (iv)  $\frac{1}{3}\tan(x^3) + c$       (v)  $-e^{\cos^2 x} + c$       (vi)  $\frac{1}{16}(x^2 - 6x + 5)^8 + c$
4. (i)  $\frac{1}{4}(\tan^{-1} x)^4 + c$       (ii)  $\frac{2}{3}(\sin x)^{3/2} + c$       (iii)  $\frac{(\log x)^{-1}}{-1}$  (or)  $\frac{-1}{\log x} + c$   
 (iv) Hint =  $y = \log(\sec x)$   
 $\log[\log(\sec x)] + c$       (v)  $\frac{1}{2}e^{2\sin^{-1} x} + c$       (vi)  $2(x^2 + 3x - 4)^{3/2} + c$
5. (i)  $\frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right) + c$       (ii)  $\frac{1}{10}\tan^{-1}\left(\frac{2x}{5}\right) + c$       (iii)  $\frac{1}{28}\log\left[\frac{2+(7x-3)}{2-(7x+3)}\right] + c$   
 (iv)  $\frac{1}{20}\log\left[\frac{(5x+2)-2}{(5x+2)+2}\right] + c$       (v)  $\sin^{-1}\left(\frac{x+1}{5}\right) + c$       (vi)  $\frac{1}{2}\sin^{-1}\left(\frac{2x}{13}\right) + c$

# UNIT – V

## INTEGRAL CALCULUS-II

### 5.1 INTEGRATION BY PARTS

Integrals of the form  $\int x \sin nx \, dx$ ,  $\int x \cos nx \, dx$ ,  $\int x e^{nx} \, dx$ ,  $\int x^n \log x \, dx$  and  $\int \log x \, dx$ . Simple Problems.

### 5.2 BERNOULLI'S FORMULA

Evaluation of the integrals  $\int x^m \sin nx \, dx$ ,  $\int x^m \cos nx \, dx$  and  $\int x^m e^{nx} \, dx$  where  $m \leq 2$  using Bernoulli's formula. Simple Problems.

### 5.3 DEFINITE INTEGRALS

Definition of definite Integral, Properties of definite Integrals – Simple Problems.

## 5.1 INTEGRATION BY PARTS

### Introduction:

When the integrand is a product of two functions and the method of decomposition or substitution cannot be applied, then the method of by parts is used. In differentiation we have seen.

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\text{i.e } d(uv) = u dv + v du$$

Integrating both sides:

$$\int d(uv) = \int u dv + \int v du$$

$$uv = \int u dv + \int v du$$

$$\text{i.e } \int u dv = uv - \int v du$$

$\therefore \int u dv = uv - \int v du$  is called Integration by parts formula.

The above formula is used by taking proper choice of 'u' and 'dv'. 'u' should be chosen based on the following order of Preference.

1. Inverse trigonometric functions
2. Logarithmic functions
3. Algebraic functions
4. Trigonometric functions
5. Exponential functions

Simply remember ILATE.

## WORKED EXAMPLES

### PART – A

1. Evaluate:  $\int x \cos x \, dx$

**Solution:**

$$\int u \, dv = uv - \int v \, du$$

choosing  $u = x$  and  $dv = \cos x \, dx$

$$du = dx \quad \int dv = \int \cos x \, dx$$

$$v = \sin x$$

$$\begin{aligned} \therefore \int x \cos x \, dx &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + c \end{aligned}$$

2. Evaluate:  $\int \log x \, dx$

**Solution:**

Choosing  $u = \log x$  and  $dv = dx$

$$\int u \, dv = uv - \int v \, du$$

$$du = \frac{1}{x} \, dx \quad \int dv = \int dx$$

$$v = x$$

$$\begin{aligned} \therefore \int \log x \, dx &= \log x \cdot x - \int x \cdot \frac{1}{x} \, dx \\ &= x \log x - \int dx \\ &= x \log x - x + c \end{aligned}$$

### PART – B & C

1. Evaluate:  $\int xe^{-x} \, dx$

**Solution:**

$$u = x \quad dv = e^{-x} \, dx$$

$$\frac{du}{dx} = 1 \quad v = -e^{-x}$$

$$du = dx$$

$$\int u \, dv = uv - \int v \, du$$

$$\begin{aligned} \therefore \int xe^{-x} \, dx &= (x)(-e^{-x}) - \int -e^{-x} \, dx \\ &= -xe^{-x} + \int e^{-x} \, dx \\ &= -xe^{-x} - e^{-x} + c \\ &= -e^{-x}(x+1) + c \end{aligned}$$

2. Evaluate:  $\int x \sin 2x \, dx$

**Solution:**

$$u = x \quad dv = \sin 2x \, dx$$

$$\frac{du}{dx} = 1 \quad v = \frac{-\cos 2x}{2}$$

$$du = dx$$

$$\int u \, dv = uv - \int v \, du$$

$$\begin{aligned} \int x \sin 2x &= (x) \left( \frac{-\cos 2x}{2} \right) - \int \frac{-\cos 2x}{2} \, dx \\ &= \frac{-x \cos 2x}{2} + \frac{1}{2} \int \cos 2x \, dx \\ &= \frac{-x \cos 2x}{2} + \frac{\sin 2x}{4} + c \end{aligned}$$

3. Evaluate:  $\int x \log x \, dx$

**Solution:**

Choosing  $u = \log x$  and  $dv = x \, dx$

$$du = \frac{1}{x} \, dx \quad v = \frac{x^2}{2}$$

$$\begin{aligned} \therefore \int x \log x \, dx &= \log x \left( \frac{x^2}{2} \right) - \int \frac{x^2}{2} \times \frac{1}{x} \, dx \\ &= \frac{x^2 \log x}{2} - \frac{1}{2} \int x \, dx \\ &= \frac{x^2 \log x}{2} - \frac{x^2}{4} + c \end{aligned}$$

## 5.2 BERNOULLI'S FORM OF INTEGRATION BY PARTS

If  $u$  and  $v$  are functions  $x$ , then Bernoulli's form of integration by parts formula is

$$\int u dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$$

Where  $u', u'', u''', \dots$  are successive differentiation of the function  $u$  and  $v, v_1, v_2, v_3, \dots$  the successive integration of the function  $dv$ .

**Note:**

The function ' $u$ ' is differentiated upto constant.

### PART B & C

**Example:**

1. Evaluate:  $\int x^2 e^{2x} dx$

**Solution:**

Choosing  $u = x^2$  and  $dv = e^{2x} dx$

$$u' = 2x \quad v = \frac{e^{2x}}{2}$$

$$u'' = 2 \quad v_1 = \frac{e^{2x}}{4}$$

$$v_2 = \frac{e^{2x}}{8}$$

$$\int u dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$$

$$\int x^2 e^{2x} dx = x^2 \frac{e^{2x}}{2} - \frac{2xe^{2x}}{4} + \frac{2e^{2x}}{8} + c$$

2. Evaluate:  $\int x^2 \sin 2x dx$

**Solution:**

$$\int u dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$$

$$u = x^2 \quad \text{and} \quad dv = \sin 2x dx$$

$$u' = 2x \quad v = \frac{-\cos 2x}{2}$$

$$u'' = 2 \quad v_1 = \frac{-\sin 2x}{4}$$

$$v_2 = \frac{\cos 2x}{8}$$

$$\therefore \int x^2 \sin 2x dx = (x^2) \left( \frac{-\cos 2x}{2} \right) - (2x) \left( \frac{-\sin 2x}{4} \right) + (2) \left( \frac{\cos 2x}{8} \right) + c$$

$$= \frac{-x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + c$$

3. Evaluate:  $\int x^2 \cos 3x \, dx$

**Solution:**

$$\int u \, dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$$

$$u = x^2 \quad \text{and} \quad dv = \cos 3x \, dx$$

$$u' = 2x \quad v = \frac{\sin 3x}{3}$$

$$u'' = 2 \quad v_1 = \frac{-\cos 3x}{9}$$

$$v_2 = \frac{-\sin 2x}{27}$$

$$\begin{aligned} \therefore \int x^2 \cos 3x \, dx &= (x^2) \left( \frac{\sin 3x}{3} \right) - (2x) \left( \frac{-\cos 3x}{9} \right) + (2) \left( \frac{-\sin 3x}{27} \right) + c \\ &= \frac{x^2 \sin 3x}{3} + \frac{2x \cos 3x}{9} - \frac{2 \sin 3x}{27} + c \end{aligned}$$

### 5.3 DEFINITE INTEGRALS

Definition of Definite Integrals:

Let  $\int f(x) \, dx = F(x) + c$ , where  $c$  is the arbitrary constant of integration. The value of the integral.

when  $x = b$ , is  $F(b) + c$  ..... (1)

and when  $x = a$ , is  $F(a) + c$  ..... (2)

Subtracting (2) from (1) we have

$F(b) - F(a) = (\text{the value of the integral when } x = b) - (\text{The value of the integral when } x = a).$

$$\begin{aligned} \text{i.e. } \int_a^b f(x) \, dx &= [F(x) + c]_a^b \\ &= [F(b) + c] - F(a) + c \\ &= F(b) + c - F(a) - c \\ &= F(b) - F(a) \end{aligned}$$

Thus

$\int_a^b f(x) \, dx$  is called the definite integral, here  $a$  and  $b$  are called the lower limit and upper limit of integral respectively.

#### Properties of Definite Integrals:

Certain properties of definite integral are useful in solving problems. Some of the often used properties are given below.

$$1. \int_a^b f(x) dx = -\int_b^a f(x) dx$$

$$2. \int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

3. If  $a < c < b$  in  $[a, b]$

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$4. \int_a^b f(x) dx = \int_a^b f(t) dt$$

$$5. \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$6. \int_{-a}^{+a} f(x) dx = 2 \int_0^a f(x) dx \quad \text{if } f(x) \text{ is even i.e. } f(-x) = f(x).$$

$$= 0 \quad \text{if } f(x) \text{ is odd i.e. } f(-x) = -f(x)$$

$$7. \int_0^{2a} f(x) dx = \int_0^{2a} f(2a-x) dx$$

## WORKED EXAMPLES

### PART – A

1. Evaluate:  $\int_1^2 \frac{1}{x} dx$

**Solution:**

$$\begin{aligned} \text{Let } I &= \int_1^2 \frac{1}{x} dx \\ &= [\log x]_1^2 \\ &= \log 2 - \log 1 \quad (\because \log 1 = 0) \\ I &= \log 2 \end{aligned}$$

2. Evaluate:  $\int_0^{\pi/2} \sin x dx$

**Solution:**

$$\begin{aligned} \text{Let } I &= \int_0^{\pi/2} \sin x dx \\ &= [-\cos x]_0^{\pi/2} \\ &= -\cos \frac{\pi}{2} + \cos 0 \\ &= 0 + 1 \\ I &= 1 \end{aligned}$$

3. Evaluate:  $\int_0^{\pi/4} \sec^2 x \, dx$

**Solution:**

$$\text{Let } I = \int_0^{\pi/4} \sec^2 x \, dx$$

$$= [\tan x]_0^{\pi/4}$$

$$= \tan \frac{\pi}{4} - \tan 0$$

$$I = 1$$

4. Evaluate:  $I = \int_1^2 (x + x^2) \, dx$

**Solution:**

$$\text{Let } I = \int_1^2 (x + x^2) \, dx$$

$$I = \left[ \frac{x^2}{2} + \frac{x^3}{3} \right]_1^2$$

$$= \left[ \left( \frac{2^2}{2} + \frac{2^3}{3} \right) - \left( \frac{1^2}{2} + \frac{1^3}{3} \right) \right]$$

$$= \left( \frac{4}{2} + \frac{8}{3} \right) - \left( \frac{1}{2} + \frac{1}{3} \right)$$

$$= \left( \frac{12+16}{6} \right) - \left( \frac{3+2}{6} \right)$$

$$= \frac{28}{6} - \frac{5}{6}$$

$$= \frac{28-5}{6}$$

$$I = \frac{23}{6}$$

5. Evaluate:  $\int_0^1 x^2(1-x)^{10} \, dx$

**Solution:**

$$\text{By property : } \int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$$

$$\int_0^1 x^2(1-x)^{10} \, dx = \int_0^1 (1-x)^2 [1-(1-x)]^{10} \, dx$$

$$= \int_0^1 (1-2x+x^2) x \, dx$$

$$= \left[ \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_0^1$$

$$= \left( \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) - 0$$

$$= \frac{6-8+3}{12} = \frac{1}{12}$$

## PART - B &amp; C

1. Evaluate:  $\int_0^{\pi/2} \frac{\cos^2 x}{1 + \sin x} dx$

**Solution:**

$$\begin{aligned} \text{Let } I &= \int_0^{\pi/2} \frac{\cos^2 x}{1 + \sin x} dx \\ &= \int_0^{\pi/2} \frac{1 - \sin^2 x}{1 + \sin x} dx \\ &= \int_0^{\pi/2} \frac{(1 - \sin x)(1 + \sin x)}{1 + \sin x} dx \\ &= \int_0^{\pi/2} (1 - \sin x) dx \\ &= [x + \cos x]_0^{\pi/2} \\ &= \left(\frac{\pi}{2} + \cos \frac{\pi}{2}\right) - (0 + \cos 0) \\ &= \left(\frac{\pi}{2} + 0\right) - (0 + 1) \\ I &= \frac{\pi}{2} - \end{aligned}$$

2. Evaluate:  $\int_0^{\pi/2} \cos^2 x dx$

**Solution:**

$$\begin{aligned} \text{Let } I &= \int_0^{\pi/2} \cos^2 x dx \\ I &= \int_0^{\pi/2} \cos^2 x dx \quad \because \cos^2 x = \frac{1 + \cos 2x}{2} \\ &= \frac{1}{2} \int_0^{\pi/2} (1 + \cos 2x) dx \\ &= \frac{1}{2} \left[ x + \frac{\sin 2x}{2} \right]_0^{\pi/2} \\ &= \frac{1}{2} \left[ \left( \frac{\pi}{2} + \frac{\sin \pi}{2} \right) - \left( 0 + \frac{\sin 0}{2} \right) \right] \\ &= \frac{1}{2} \left[ \frac{\pi}{2} + 0 - 0 + 0 \right] \\ I &= \frac{1}{2} \left[ \frac{\pi}{2} \right] \\ I &= \frac{\pi}{4} \end{aligned}$$

3. Evaluate:  $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$

**Solution:**

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx \text{ by property}$$

$$\therefore \text{ Let } I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx \quad \dots\dots(1)$$

$$= \int_0^{\pi/2} \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)} dx$$

$$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \quad \dots\dots(2)$$

Adding (1) & (2)

$$2I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx$$

$$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx$$

$$= \int_0^{\pi/2} dx$$

$$2I = [x]_0^{\pi/2}$$

$$2I = \frac{\pi}{2} - 0$$

$$I = \frac{\pi}{4}$$

4. Evaluate:  $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x dx$

**Solution:**

The given Integrand  $\sin^2 x \cos x$  is even.

$$\therefore \int_{-\pi/2}^{\pi/2} \sin^2 x \cos x dx = 2 \int_0^{\pi/2} \sin^2 x \cos x dx$$

$$= 2 \left[ \frac{\sin^3 x}{3} \right]_0^{\pi/2}$$

$$= \frac{2}{3} \left[ \sin^3 \frac{\pi}{2} - \sin^3 0 \right]$$

$$= \frac{2}{3} (1)$$

$$= \frac{2}{3}$$

$$f(x) = \sin x$$

$$f'(x) = \cos x$$

$$n = 2$$

$$\text{using } \int [f(x)]^n f'(x) dx$$

$$= \frac{[f(x)]^{n+1}}{n+1} + c$$

**EXERCISE****PART – A**

1. Evaluate:  $\int x e^x dx$
2. Evaluate:  $\int x \sin x dx$
3. Evaluate:  $\int x e^{2x} dx$
4. Evaluate:  $\int x \cos x dx$

**PART – B**

1. Evaluate:  $\int x^2 \sin x dx$
2. Evaluate:  $\int x^2 \cos x dx$
3. Evaluate:  $\int x^2 e^x dx$
4. Evaluate:  $\int \log x dx$
5. Evaluate:  $\int_0^2 (x + x^2) dx$
6. Evaluate:  $\int_0^{\pi/2} \cos x dx$
7. Evaluate:  $\int_0^{\pi/2} \sin x dx$
8.  $\int_0^{\pi} \cos x dx$

**PART – C**

Evaluate the following:

- |  |  |
|--|--|
| 1) $\int x^3 e^x dx$                               | 2) $\int x^3 \sin x dx$                  |
| 3) $\int x^3 \cos x dx$                            | 4) $\int \log x \cdot x^n dx$            |
| 5) $\int_0^{\pi/2} \sin^2 x dx$                    | 6) $\int_0^{\pi/2} \cos^2 x dx$          |
| 7) $\int_0^1 (2x+3)^4 dx$                          | 8) $\int_0^{\pi/6} 2 \sin 3x \cos 2x dx$ |
| 9) $\int_0^{\pi/2} \frac{\sin^2 x}{1 - \cos x} dx$ | 10) $\int_0^{\pi/2} \sin^7 x \cos x dx$  |

**ANSWERS****PART – A**

- |  |                             |
|--|-----------------------------|
| 1) $x e^x - e^x + c$                           | 2) $-x \cos x + \sin x + c$ |
| 3) $x \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + c$ | 4) $x \sin x + \sin x + c$  |

**PART – B**

1)  $-x^2 \cos x + 2x \sin x + 2 \cos x + c$

2)  $x^2 \sin x + 2x \cos x - 2 \sin x + c$

3)  $x^2 e^x - 2x e^x + 2e^x + c$

4)  $x \log x - x + c$

5)  $\frac{23}{6}$

6) 1

7) 1

8) 0

**PART – C**

1)  $x^3 e^x - 3x^2 e^x + 6x e^x - 6 e^x + c$

2)  $-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + c$

3)  $x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + c$

4)  $\frac{x^{n+1} \log x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + c$

5)  $\frac{\pi}{4}$

6)  $\frac{\pi}{4}$

7) 288.2

8)  $\frac{6}{5}$

9)  $\frac{\pi}{2} + 1$

10)  $\frac{1}{8}$

**APPLIED  
MATHEMATICS**

# ***NOTES***

# UNIT – I

## PROBABILITY DISTRIBUTION-I

### 1.1 RANDOM VARIABLE

Definition of Random variable – Types –Probability function – Probability density function. Simple Problems.

### 1.2 MATHEMATICAL EXPECTATION

Mathematical Expectation of discrete random variable, mean and variance. Simple Problems.

### 1.3 BINOMIAL DISTRIBUTION

Definition Binomial distribution  $P(X = x) = {}^n C_x p^x q^{n-x}$  where  $x = 0, 1, 2, \dots$  Statement only. Expression for mean and variance. Simple Problems.

## 1.1 RANDOM VARIABLE

### Introduction:

Let a coin be tossed. Nobody knows what we will get whether a head or tail. But it is certain that either a head or tail will occur. In a similar way if a dice is thrown. We may get any of the faces 1, 2, 3, 4, 5 and 6. But nobody knows which one will occur. Experiments of this type where the outcome cannot be predicted are called "random" experiments.

The word probability or chance is used commonly in day-to-day life. For example the chances of India and South Africa winning the world cup Cricket, before the start of the game are equal (i.e., 50:50). We often say that it will rain tomorrow. Probably I will not come to function today. All these terms-chance, probable etc., convey the same meaning i.e., that event is not certain to take place. In other words there is an uncertainty about the happening of the event. The term probability refers to the randomness and uncertainty.

### Trail and Event:

Consider an experiment of throwing a coin. When tossing a coin, we may get a head (H) or tail (T). Here tossing of a coin is a trail and getting a head or tail is an event.

From a pack of cards drawing any three cards is trail and getting a King or queen or a jack are events.

Throwing of a dice is a trail and getting 1 or 2 or 3 or 4 or 5 or 6 is an event.

### Sample Space:

The set of all possible cases of an experiment is called the sample space and is denoted by S.

### Mathematical Definition of Probability:

$$\text{Probability of Event } E = \frac{\text{No. of favourable cases}}{\text{Total no. of outcomes}}$$

$$P(E) = \frac{m}{n}$$

Where 'm' is number of favourable cases = n (E) and 'n' is number of exhaustive cases = n(S).

### Random Variable:

A function X which transforms events of a random experiment into real numbers is called random variable. It is denoted as  $X : S \rightarrow R$ , where S is sample space of random experiment and R is set of real numbers.

### Example:

Two coins are tossed at a time.

Sample space is  $S = \{HH, HT, TH, TT\}$ . If we take X is the number of heads appearing then HH becomes 2. HT and TH becomes 1, and TT becomes 0.

$\therefore$  X (number of heads) is a random variable.

### Types of Random Variables:

There are two types of random variables known as

- (i) Discrete random variable
- (ii) Continuous random variable

### Discrete random variable:

If a random variable takes only a finite or a countable number of values, it is called a discrete random variable.

For example, when two coins are tossed the number of heads obtained is the random variable X. Where X assumes the values 0, 1, 2. Which is a countable set. Such a variable is called discrete random variable.

### Definition:

### Probability Mass Function:

Let X be a discrete random variable with values  $x_1, x_2, x_3, \dots, x_n$ . Let  $p(x_i)$  be a number associated with each  $x_i$ .

Then the function p is called the probability mass function of X if it satisfies the conditions

- (i)  $p(x_i) \geq 0$  for  $i = 1, 2, 3, \dots, n$
- (ii)  $\sum p(x_i) = 1$

The set of ordered pairs  $[x_i, p(x_i)]$  is called the probability distribution of X.

### Continuous Random Variable:

A random variable X is said to be continuous if it can take all possible values between certain limits.

### Examples:

1. Life time of electric bulb in hours.
2. Height, weight, temperature, etc.

### Definition:

### Probability density function:

A function  $f$  is said to be probability density function (pdf) of the continuous random variable X if it satisfies the following condition:

1.  $f(x) \geq 0$  for all  $x \in R$

2.  $\int_{-\infty}^{+\infty} f(x) dx = 1$

**Definition:**

Distribution function (Cumulative Distribution Function).

The function  $F(X)$  is said to be the distribution function of the random variable  $X$ , if

$$F(X) = P(X \leq x); \quad -\infty \leq x \leq \infty$$

The distribution function  $F$  is also called cumulative distribution function.

**Note:**

1. If  $X$  is a discrete random variable then from the definition it follows that  $F(X) = \sum p(x_i)$ , where the summation is overall  $x_i$ , such that  $x_i \leq x$ .
2. If  $X$  is a continuous random variable, then from the definition it follows that

$$y = Ae^x + Be^{2x} + \frac{e^{-4x}}{10} + 4xe^{2x},$$

where  $f(t)$  the value of the probability density function of  $X$  at  $t$ .

**WORKED EXAMPLES****PART – A**

1. Find the probability distribution of  $X$  when tossing a coin, when  $X$  is defined as getting a head.

**Solution:**

Let  $X$  denote getting a head.

$$\text{Probability of getting a head} = \frac{1}{2}$$

$$\text{Probability of getting a tail} = \frac{1}{2}$$

The probability distribution of  $X$  is given by

$X$	:	0	1
$P(X = x)$ :		$\frac{1}{2}$	$\frac{1}{2}$

2. When throwing a die what is probability of getting a 4?

**Solution:**

Total number of cases  $n = 6$  (1, 2, 3, 4, 5, 6)

Number of favourable cases = 1

$$\therefore \text{Probability of getting 4} = \frac{m}{n} = \frac{1}{6}$$

3. Verify that  $f(x) = \begin{cases} \frac{2x}{9}, & 0 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$  is a probability density function.

**Solution:**

$$\int_{-\infty}^{+\infty} f(x) dx = \int_0^3 \frac{2x}{9} dx = \frac{2}{9} \left[ \frac{x^2}{2} \right]_0^3$$

$$= \frac{2}{9} \left[ \frac{9}{2} \right]$$

$$= 1$$

$\Rightarrow f(x)$  is a probability density function.

**PART – B**

1. A random variable X has the following probability distribution:

X	:	0	1	2	3	4
P(X)	:	3a	4a	6a	7a	8a

Find (i) Value of a.

(ii)  $P(X \leq 2)$

**Solution:**

(i) Since  $\sum P(X) = 1$

$$\Rightarrow 3a + 4a + 6a + 7a + 8a = 1$$

$$28a = 1$$

$$a = \frac{1}{28}$$

(ii)  $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= 3a + 4a + 6a$$

$$= 13a$$

$$= 13 \left( \frac{1}{28} \right)$$

$$= \frac{13}{28}$$

**PART – C**

1. If  $f(x) = \begin{cases} kx^2, & 0 \leq x \leq 3 \\ 0 & \text{else where} \end{cases}$ , is a pdf, find the value of k.

**Solution:**

Since  $f(x)$  is a pdf.

We have

$$\int_0^3 f(x) dx = 1$$

$$\int_{-\infty}^{+\infty} kx^2 dx = 1$$

$$K \left[ \frac{x^3}{3} \right]_0^3 = 1$$

$$K \left[ \frac{3^3}{3} - \frac{0^3}{3} \right] = 1$$

$$K \left[ \frac{27}{3} \right] = 1$$

$$9K = 1$$

$$K = \frac{1}{9}$$

## 1.2 MATHEMATICAL EXPECTATION OF DISCRETE VARIABLE

### Expectation of a discrete random variable:

If  $X$  denotes a discrete random variable which can assume the value  $x_1, x_2, \dots, x_n$  with respective probabilities  $p_1, p_2, \dots, p_n$  then the mathematical expectation of  $X$ , denoted by  $E(X)$  is defined by

$$E(X) = p_1x_1 + p_2x_2 + \dots + p_nx_n \\ = \sum_{i=1}^n p_i x_i \quad \text{where } \sum_{i=1}^n p_i = 1$$

Thus  $E(X)$  is the weighted arithmetic mean of the values  $x_i$  with the weight to  $p(x_i)$ .

$\therefore$  Mean  $\bar{X} = E(X)$ .

Hence the mathematical expectation  $E(X)$  of a random variable is simply the arithmetic mean.

Result: If  $\phi(x)$  is a function of a random variable  $X$ , then

$$E[\phi(x)] = \sum P(X = x) \phi(x).$$

$$E(X^2) = p_1x_1^2 + p_2x_2^2 + p_3x_3^2 + \dots$$

### Properties of mathematical expectation:

1.  $E(C) = C$ , where  $C$  is a constant.
2.  $E(CX) = CE(X)$ .
3.  $E(ax + b) = aE(X) + b$ , where  $a, b$  are constants.
4. Variance of  $X = \text{Var}(X) = E\{X - E(X)\}^2$ .
5.  $\text{Var}(X) = E(X^2) - [E(X)]^2$
6.  $\text{Var}(X \pm C) = \text{Var}(X)$  where  $C$  is a constant.
7.  $\text{Var}(aX) = a^2\text{Var}(X)$ .
8.  $\text{Var}(aX + b) = a^2\text{Var}(X)$ .
9.  $\text{Var}(C) = 0$ , where  $C$  is a constant.

### WORKED EXAMPLES

#### PART – A

1. Find the expected value of the random variable  $X$  has the following probability distribution.

$X$	:	1	2	3
$P(X)$	:	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$

#### **Solution:**

Expectation of  $X$ .

$$E(X) = \sum x_i P(X_i) \\ = 1 \times \frac{1}{6} + 2 \times \frac{4}{6} + 3 \times \frac{1}{6} \\ = \frac{1}{6} + \frac{8}{6} + \frac{3}{6} \\ = \frac{1+8+3}{6}$$

$$E(X) = \frac{12}{6} = 2$$

2. Evaluate  $\text{Var}(2X \pm 3)$ .

**Solution:**

We have  $\text{Var}(ax \pm b) = a^2 \text{Var}(X)$

$$\begin{aligned}\text{Var}(2X \pm 3) &= 2^2 \text{Var}(X) \\ &= 4 \text{Var}(X)\end{aligned}$$

3. A random variable  $X$  has  $E(X) = 2$  and  $E(X^2) = 8$ . Find its variance.

**Solution:**

$$\begin{aligned}\text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= 8 - [2]^2 \\ &= 8 - 4\end{aligned}$$

$$\text{Var}(X) = 4$$

### PART – B

1. A random variable  $X$  has the following probability distribution:

$X$	:	$-3$	$6$	$9$
$P(X)$	:	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

Find  $E(2X + 1)$ .

**Solution:**

$$\begin{aligned}E(2X + 1) \\ &= E(2X) + E(1)\end{aligned}$$

$$E(2x + 1) = 2E(X) + 1$$

Where  $E(X) = \sum x_i P(x_i)$

$$= (-3)\left(\frac{1}{6}\right) + 6\left(\frac{1}{2}\right) + 9\left(\frac{1}{3}\right)$$

$$= -\frac{3}{6} + \frac{6}{2} + \frac{9}{3}$$

$$= \frac{-3 + 18 + 18}{6}$$

$$= \frac{-3 + 36}{6} = \frac{33}{6} = \frac{11}{2}$$

$$E(2X + 1) = 2\left(\frac{11}{2}\right) + 1$$

$$= 11 + 1$$

$$E(2X + 1) = 12$$

**PART – C**

1. The monthly demand for ladies hand bags is known to have the following distribution.

Demand	1	2	3	4	5	6
Probability	0.1	0.15	0.20	0.25	0.18	0.12

Determine the expected demand for ladies hand bags. Also obtain variance.

**Solution:**

$(x^2)$	1	4	9	16	25	36
Demand (x)	1	2	3	4	5	6
Probability P(x)	0.1	0.15	0.20	0.25	0.18	0.12

$$\text{Expected demand, } E(x) = \sum_{i=1}^n x_i p_i$$

$$\begin{aligned} &= 1(0.1) + 2(0.15) + 3(0.20) + 4(0.25) + 5(0.18) + 6(0.12) \\ &= 0.1 + 0.3 + 0.6 + 1.0 + 0.9 + 0.72 \\ &= 3.62 \end{aligned}$$

$$E(x^2) = \sum_{i=1}^n x_i^2 p_i^2$$

$$\begin{aligned} &= 1(0.1) + 4(0.15) + 9(0.2) + 16(0.25) + 25(0.18) + 35(0.12) \\ &= 0.1 + 0.6 + 1.8 + 4.0 + 4.5 + 4.32 \\ &= 15.32 \end{aligned}$$

$$\begin{aligned} \therefore \text{Variance} &= E(x^2) - [E(x)]^2 \\ &= 15.32 - (3.62)^2 \\ &= 15.32 - 13.10 \\ &= 2.22 \end{aligned}$$

## 1.3 BINOMIAL DISTRIBUTION

### Introduction:

Binomial distribution was discovered by James Bernoulli (1654 – 1705) in the year 1700 and was first published in 1713.

An experiment which has two mutually disjoint outcomes is called a Bernoulli trial. The two outcomes are usually called "success" and "failure".

An experiment consisting of repeated number of Bernoulli trials is called Binomial experiment. A Binomial distribution can be used under the following conditions.

- (i) The number of trials is finite.
- (ii) The trials are independent of each other.
- (iii) The probability of success is constant for each trial.

### Probability function of Binomial Distribution:

Let  $X$  denotes the number of success in  $n$  trial satisfying binomial distribution conditions.  $X$  is a random variable which can take the values  $0, 1, 2, \dots, n$ , since we get no success, one success or all  $n$  success.

The general expression for the probability of  $x$  success is given by

$$P(X = x) = {}^n C_x p^x q^{n-x}, \quad x = 0, 1, 2, 3, \dots, n.$$

Where  $P$  = probability of success. In each trial and  $q = 1 - p$ .

### Definition:

A random variable  $X$  is said to follow binomial distribution, if the probability mass function is given by

$$P(X = x) = \begin{cases} {}^n C_x p^x q^{n-x}, & x = 0, 1, 2, 3, \dots, n. \\ 0 & \text{otherwise} \end{cases}$$

Where  $n, p$  are called parameter of the distribution.

### Constants of the binomial distribution:

$$\text{Mean} = np$$

$$\text{Variance} = npq$$

$$\text{Standard Deviation} = \sqrt{npq}$$

### Note:

- (i)  $0 \leq p \leq 1, 0 \leq q \leq 1$  and  $p + q = 1$ .
- (ii) In binomial distribution mean is always greater than variance.
- (iii) To denote the random variable  $X$  which follows binomial distribution with parameters  $n$  and  $p$  is  $X \sim B(n, p)$ .

## WORKED EXAMPLES

### PART – A

1. Find  $n$  and  $p$  in the binomial distribution whose mean is 3 and variance is 2.

**Solution:**

Given, mean = 3

i.e  $np = 3$  and Variance  $npq = 2$

$$\frac{npq}{np} = \frac{2}{3}$$

$$q = \frac{2}{3} \quad \therefore p = 1 - q$$

$$p = 1 - \frac{2}{3}$$

$$p = \frac{1}{3}$$

2. In a binomial distribution if  $n = 9$  and  $p = \frac{1}{3}$ , what is the value of variance?

**Solution:**

Given:  $n = 9$ ,  $p = \frac{1}{3}$

$$q = 1 - p$$

$$q = 1 - \frac{1}{3}$$

$$q = \frac{2}{3}$$

$\therefore$  Variance =  $npq$

$$= 9 \times \frac{1}{3} \times \frac{2}{3}$$

$$\text{Variance} = 2$$

### PART – B

1. If the sum of mean and variance of a binomial distribution is 4.8 for 5 trials, find the distribution.

**Solution:**

Mean =  $np$  ; Variance =  $npq$

Given sum of mean and variance is 4.8.

$$\text{i.e } np + npq = 4.8$$

$$np(1 + q) = 4.8$$

$$np(1 + 1 - p) = 4.8$$

$$5p(2 - p) = 4.8$$

$$10p - 5p^2 - 4.8 = 0$$

$$\text{or } 5p^2 - 10p + 4.8 = 0 \quad \Rightarrow p = 1.2, 0.8$$

$\therefore p = 0.8$ ,  $q = 0.2$  ( $\because p$  cannot be greater than 1)

The binomial distribution is

$$P(X = x) = {}^5C_x (0.8)^x (0.2)^{5-x}$$

Where  $x = 0, 1, 2, 3, 4, 5$

2. In tossing 10 coin. What is the chance of having exactly 5 heads.

**Solution:**

Let X denote number of heads

$$p = \text{Probability of getting a head} = \frac{1}{2}$$

$$q = 1 - p$$

$$= 1 - \frac{1}{2}$$

$$q = \frac{1}{2}$$

n = number of trial is 10.

∴ The binomial distribution is

$$P(X = x) = {}^n C_x p^x q^{n-x}$$

Probability of getting exactly 5 heads

$$\begin{aligned} P(X = 5) &= {}^{10} C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{10-5} \\ &= {}^{10} C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^5 \\ &= {}^{10} C_5 \left(\frac{1}{2}\right)^{10} \\ &= \frac{63}{256} \end{aligned}$$

### PART – C

1. Ten coins are tossed simultaneously find the probability of getting atleast seven heads.

**Solution:**

$$\text{Given } n = 10, \quad p = \frac{1}{2}, \quad q = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(X = x) = {}^n C_x p^x q^{n-x}$$

$$= {}^{10} C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x}$$

$$= {}^{10} C_x \left(\frac{1}{2}\right)^{x+10-x}$$

$$P(X = x) = {}^{10} C_x \left(\frac{1}{2}\right)^{10}$$

Probability of getting atleast seven heads is

$$\begin{aligned} P(x \geq 7) &= P(x = 7) + P(x = 8) + P(x = 9) + P(x = 10) \\ &= {}^{10} C_7 \left(\frac{1}{2}\right)^{10} + {}^{10} C_8 \left(\frac{1}{2}\right)^{10} + {}^{10} C_9 \left(\frac{1}{2}\right)^{10} + {}^{10} C_{10} \left(\frac{1}{2}\right)^{10} \\ &= \left(\frac{1}{2}\right)^{10} [{}^{10} C_7 + {}^{10} C_8 + {}^{10} C_9 + {}^{10} C_{10}] \\ &= \frac{1}{2^{10}} [120 + 45 + 10 + 1] \\ &= \frac{176}{1024} \\ P(x \geq 7) &= \frac{11}{64} \end{aligned}$$

**EXERCISE****PART – A**

- When throwing a die what is probability of getting a 4?
- Find the chance that if a card is drawn at random from an ordinary pack, it is one of the jacks.
- A bag contains 7 white and 9 red balls. Find the probability of drawing a white ball.
- A random variable X has the following distribution :

X	0	1	2	3
P(X = x)	3a	4a	6a	7a

Find the value of 'a'.

- If  $E(X) = 8$  what is the value of  $E(3X)$ .
- If  $V(X) = 2$ , what is the value of  $V(5X + 7)$ .
- Find the mean of the binomial distribution if  $P(x) = 20C_x \left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{20-x}$ .

**PART – B**

- A random variable X has the following distribution function.

X	0	1	2	3
P(X = x)	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

Find  $P(X \leq 2)$ .

- Show that  $\begin{cases} f(x) = 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$  is a probability density function.
- A random variable X has  $E(X) = 2$  and  $E(X^2) = 8$  find the variance.
- Given  $E(X + C) = 8$  and  $E(X - C) = 12$  find the value of C.
- A random variable X has the following probability distribution

X	0	1	2	3
P(X)	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$

Find  $E(X^2)$ .

- Write down the Binomial distribution for which  $n = 10$  and  $P = \frac{1}{3}$ .
- Find the variance of the binomial distribution if  $P(X = x) = 15C_x \left(\frac{3}{5}\right)^x \left(\frac{2}{5}\right)^{10-x}$ .

**PART – C**

1. A random variable  $x$  has the following probability distribution function

X	0	1	2	3	4	5
P(X = x)	a	3a	5a	7a	9a	11a

Find (i) The value of 'a' (ii)  $P(X < 4)$

2. If a random variable  $X$  has the following probability distribution

X	-1	0	1	2
P(X = x)	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$

Find (i)  $E(X)$  (ii)  $E(X^2)$  & (iii) Variance ( $X$ ).

3. Ten coins are tossed simultaneously. Find the probability of getting exactly 2 heads.  
 4. With usual notation find 'p' for the binomial distribution  $X$  if  $n = 6$  and if  $9P(X = 4) = P(X = 2)$ .  
 5. The mean and variance of a binomial variate  $X$  with parameters 'n' and  $p$  are 16 and 8 respectively. Find  $P(X = 0)$  and  $P(X = 1)$ .  
 6. Eight coins are tossed simultaneously. Find the probability of getting at least five heads.  
 7. Four coins are tossed simultaneously. Find the probability of getting (i) at least two heads (ii) at least one head.  
 8. Six coins are tossed simultaneously. Find the probability of getting at most four heads.

**ANSWERS****PART – A**

- 1)  $\frac{1}{6}$       2)  $\frac{1}{13}$       3)  $\frac{7}{16}$       4)  $\frac{1}{20}$       5) 24      6) 50      7) 8

**PART – B**

- 1)  $\frac{5}{6}$       3) 4      4)  $C = -2$       5)  $\frac{23}{6}$       6)  $10C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{10-x}$   
 7)  $15 \binom{3}{5} \binom{2}{5}$

**PART – C**

- 1)  $a = \frac{1}{36}; \frac{4}{9}$       2)  $\frac{1}{2}, \frac{11}{6}, \frac{19}{2}$       3)  $10C_2 \left(\frac{1}{2}\right)^{10}$       4)  $P = \frac{1}{4}$   
 5)  $\frac{1}{2^{32}}; \frac{1}{2^{27}}$       6)  $\frac{93}{256}$       7) (i)  $\frac{11}{16}$       (ii)  $\frac{15}{16}$       8)  $\frac{57}{64}$

## UNIT – II

# PROBABILITY DISTRIBUTION-II

### 2.1 Poisson Distribution:

Definition of poisson distribution  $P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$  where  $x = 0, 1, 2, \dots$  (statement only). Expressions of mean and variance. Simple problems.

### 2.2 Normal Distribution:

Definition of Normal and Standard normal distribution-statement only. Constants of normal distribution (results only). Properties of normal distribution-Simple problems using the table of standard normal distribution.

### 2.3 Curve fitting:

Fitting of straight line using least square method (results only simple problems).

## 2.1 POISSON DISTRIBUTION

Poisson distribution was named after the french mathematician Simeon Devis poisson who discovered it. Poisson distribution is a discrete distribution.

Poisson distribution is a limiting case of binomial distribution under the following conditions.

- (i)  $n$ , the number of Independent trials is indefinitely large. i.e  $n \rightarrow \infty$ .
- (ii)  $p$ , the constant probability of success in each trial is very small ie  $p \rightarrow 0$ .
- (iii)  $np = \lambda$  is finite where  $\lambda$  is a positive real number.

#### Definition:

A discrete random variable  $X$  is said to have a poisson distribution if the probability mass function of  $X$  is  $P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$ ,  $x = 0, 1, 2, \dots$  for some  $\lambda > 0$  and  $\lambda$  is the parameter of the poisson distribution.

#### Constants of poisson distribution:

- (i) Mean =  $\lambda$
- (ii) Variance =  $\lambda$
- (iii) Standard deviation =  $\sqrt{\lambda}$

#### Examples of poisson distribution:

1. The number of printing mistakes in each page of a book.
2. The number of suicides reported in a particular city in a particular month.
3. The number of road accidents at a particular junction per month.
4. The number of child born blind per year in a hospital.

**WORKED EXAMPLES****PART – A**

1. If the mean of the poisson distribution is 2. Find  $P(x = 0)$ .

**Solution:**

$$\text{We know } P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$\text{Given : } \lambda = 2$$

$$\begin{aligned} \therefore P(X = 0) &= \frac{e^{-2} 2^0}{0!} \\ &= e^{-2} \end{aligned}$$

2. If the mean of the poisson distribution is 4. Find the value of the variance and standard deviation.

**Solution:**

$$\text{Given Mean} = \lambda = 4$$

$$\text{Variance} = \lambda = 4$$

$$\text{S.D} = \sqrt{\lambda} = \sqrt{4} = 2$$

3. In a poisson distribution if the variance is 2. and  $P = \frac{1}{100}$  what is n.

**Solution:**

$$\text{We know } \lambda = np$$

$$\text{Given: } \lambda = 2, P = \frac{1}{100}$$

$$\therefore 2 = n \frac{1}{100}$$

$$\text{i.e } n = 200$$

**PART – B**

1. In a poisson distribution if  $P(x = 3) = P(x = 2)$  find  $P(x = 0)$ .

**Solution:**

$$\text{Given: } P(x = 3) = P(x = 2)$$

$$\text{We know } P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$\therefore \frac{e^{-\lambda} \lambda^3}{3!} = \frac{e^{-\lambda} \lambda^2}{2!}$$

$$\frac{\lambda}{1 \times 2 \times 3} = \frac{1}{1 \times 2}$$

$$\lambda = 3$$

$$\begin{aligned} \therefore P(x = 0) &= \frac{e^{-3} 3^0}{0!} \\ &= e^{-3} \end{aligned}$$

2. If 10% of the screws produced by an automobile machine are defective. Find the probability that out of 15 screws selected at random only 3 are defective.

**Solution:**

Let  $p$ -denote the probability of defective screw.

$$\text{Given: } p = 10\% = \frac{10}{100} = \frac{1}{10}, n = 15$$

$$\lambda = np$$

$$= 15 \times \frac{1}{10}$$

$$= \frac{3}{2}$$

$X$ -denote the number of defective screws.

$$P(X=3) = \frac{e^{-1.5}(1.5)^3}{3!} \quad \left| \quad P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}\right.$$

$$= 0.1255$$

Probability for 3 screws to be defective = 0.1255

### PART – C

1. If  $X$  is a poisson variable with  $P(X=2) = \frac{2}{3} P(X=1)$ , find  $P(X=3)$  and  $P(X=0)$ .

**Solution:**

$$\text{Given: } P(X=2) = \frac{2}{3} P(X=1)$$

$$\text{We know } P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\text{i.e. } \frac{e^{-\lambda}\lambda^2}{2!} = \frac{2}{3} \frac{e^{-\lambda}\lambda^1}{1!}$$

$$\frac{\lambda}{1 \times 2} = \frac{2}{3} \cdot \frac{1}{1}$$

$$\lambda = \frac{4}{3}$$

$$\therefore P(X=3) = \frac{e^{-4/3} \left(\frac{4}{3}\right)^3}{3!}$$

$$= \frac{4^3 e^{-4/3}}{3^3 \cdot 3!}$$

$$= \frac{64}{27 \times 6} e^{-4/3}$$

$$\therefore P(X=3) = \frac{32}{81} e^{-4/3}$$

$$= 0.0998$$

$$P(X=0) = \frac{e^{-4/3} \left(\frac{4}{3}\right)^0}{0!}$$

$$= e^{-4/3}$$

$$= 0.2725$$

2. The probability that an electric bulb to be defective from a manufacturing unit is 0.02. In a box of 200 electric bulbs find the probability that
- Exactly 4 bulbs are defective
  - More than 3 bulbs are defective.

**Solution:**

Let  $p$ -denote the probability of one bulb to be defective.

$$\text{Given: } p = 0.02$$

$$n = 200$$

$$\lambda = np$$

$$= 200 \times 0.02$$

$$= 4$$

$X$ -denote number of defective bulbs.

$$\begin{aligned} \text{(i) } P(X=4) &= \frac{e^{-4}(4)^4}{4!} & \left| \quad P(X=x) &= \frac{e^{-\lambda}\lambda^x}{x!} \right. \\ &= \frac{e^{-4} \times 256}{24} \end{aligned}$$

Probability for 4 bulbs to be defective = 0.1952

$$\text{(ii) } P(X > 3) = 1 - (P \leq 3)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$$

$$= 1 - \left[ \frac{e^{-4}(4)^0}{0!} + \frac{e^{-4}(4)^1}{1!} + \frac{e^{-4}(4)^2}{2!} + \frac{e^{-4}(4)^3}{3!} \right]$$

$$= 1 - e^{-4} \left[ 1 + 4 + \frac{16}{2} + \frac{64}{6} \right]$$

$$= 1 - e^{-4} \left[ \frac{142}{6} \right]$$

$$= 1 - 0.0183 \times 23.67$$

$$= 1 - 0.4332$$

$$= 0.5668$$

Probability for more than 3 bulbs = 0.5668

## 2.2 NORMAL DISTRIBUTION

Normal distribution is a continuous distribution. Like poisson distribution, the normal distribution is obtained as a limiting case of Binomial distribution. This is the most important probability model in statistical analysis.

### Definition:

A random variable  $X$  is normally distributed with parameters  $\mu$  (called Mean) and  $\sigma^2$  (called Variance) if its p.d.f (Probability density function) is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$$

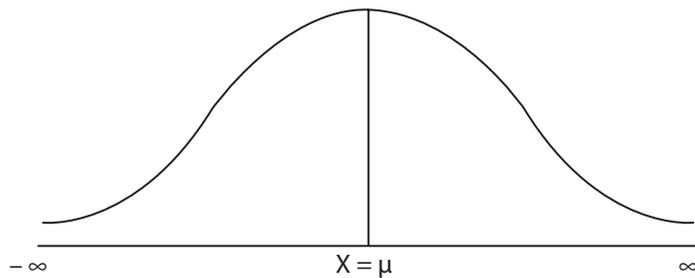
### Note:

$X \sim N(\mu, \sigma)$  denotes that the random variable  $X$  follows normal distribution with mean  $\mu$  and standard deviation  $\sigma$ .

### Constants of Normal distribution:

- (i) Mean =  $\mu$
- (ii) Variance =  $\sigma^2$
- (iii) Standard deviation =  $\sigma$

### Properties of Normal Probability curve and Normal distribution:



1. The normal curve is bell shaped.
2. The curve is symmetrical about  $X = \mu$ .
3. Mean = Median = Mode =  $\mu$ .
4. The curve attains its maximum value at  $X = \mu$  and the maximum value is  $\frac{1}{\sigma\sqrt{2\pi}}$ .
5. The normal curve is asymptotic to the x-axis.
6. The points of inflection are at  $X = \mu \pm \sigma$ .
7. Since the curve is symmetrical about  $X = \mu$ , the skewness is zero.
8. A normal distribution is a close approximation to the Binomial distribution when  $n$  is very large and  $p$  is close to  $\frac{1}{2}$ .
9. Normal distribution is also a limiting form of poisson distribution when  $\lambda \rightarrow \infty$ .

### Standard normal distribution:

A random variable  $Z$  is called a standard normal variate if its mean is zero and standard deviation is 1 and is denoted by  $N(0, 1)$ .

where  $Z = \frac{X - \mu}{\sigma}$ ,  $X$  – normal variate.

The probability density function of the standard normal variate  $Z$  is

$$\phi(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2}, \quad -\infty < Z < \infty$$

**Note:**

1. The total area under normal probability curve is unity.

$$\text{i.e. } \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} \phi(z) dz = 1$$

$$\int_{-\infty}^0 \phi(z) dx = \int_0^{\infty} \phi(z) dz = 0.5$$

(Since the normal curve is symmetrical about  $z = 0$ )

2.  $\int_0^{z_1} \phi(x) dx$  is known as normal probability Integral and gives the area under standard normal curve between the ordinates at  $z = 0$  and  $z = z_1$ .
3. In the normal probability tables, the areas are given under standard normal curve. We shall deal with the standard normal variate  $z$  rather than the variable  $X$  itself.

## WORKED EXAMPLES

### PART – A

1. When  $X = 100$ ,  $\mu = 80$ ,  $\sigma = 10$  what is the value of  $z$ ?

**Solution:**

$$\begin{aligned} \text{We know } Z &= \frac{X - \mu}{\sigma} \\ &= \frac{100 - 80}{10} \\ &= 2 \end{aligned}$$

2. If  $X$  is normally distributed with mean 80 and standard deviation 10. Express  $P(70 \leq X \leq 100)$  in terms of standard normal variate.

**Solution:**

Given: Mean =  $\mu = 80$

S.D =  $\sigma = 10$

$$\text{We know } Z = \frac{X - \mu}{\sigma}$$

$$\text{When } X = 70, Z = \frac{70 - 80}{10} = -1$$

$$\text{When } X = 100, Z = \frac{100 - 80}{10} = 2$$

$$\therefore P(70 \leq X \leq 100) = P(-1 \leq Z \leq 2)$$

### PART – B

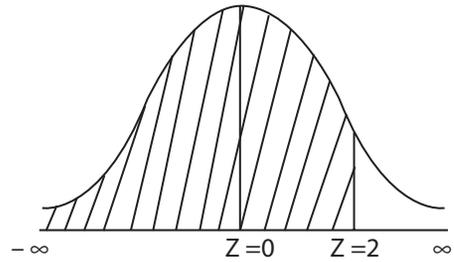
1. If  $X$  is a normal variate with mean 80 and S.D 10. Compute  $P(X \leq 100)$ .

**Solution:**

$$\text{We know } Z = \frac{X - \mu}{\sigma}$$

When  $X = 100$ ,

$$\begin{aligned} Z &= \frac{100 - 80}{10} \\ &= 2 \end{aligned}$$



$$\begin{aligned} \therefore P(X \leq 100) &= P(Z \leq 2) \\ &= P(-\infty < Z < 0) + P(0 < Z < 2) \\ &= 0.5 + 0.4772 \text{ (from table)} \\ &= 0.9772 \end{aligned}$$

2. If  $X$  is normally distributed with mean 6 and standard deviation 5. Find  $P(0 \leq X \leq 8)$ .

**Solution:**

Given: Mean  $= \mu = 6$

and S.D  $= \sigma = 5$

$$\text{We know } Z = \frac{X - \mu}{\sigma}$$

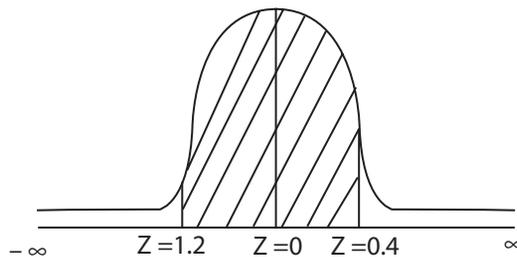
When  $X = 0$

$$Z = \frac{0 - 6}{5} = -1.2$$

When  $X = 8$

$$Z = \frac{8 - 6}{5} = \frac{2}{5} = 0.4$$

$$\begin{aligned} \therefore P(0 \leq X \leq 8) &= P(-1.2 \leq Z \leq 0.4) \\ &= P(-1.2 \leq Z \leq 0) + P(0 \leq Z \leq 0.4) \\ &= P(0 \leq Z \leq 1.2) + P(0 \leq Z \leq 0.4) \\ &\quad (\because \text{the curve is symmetry}) \\ &= 0.3849 + 0.1554 \text{ (from the table)} \\ &= 0.5403 \end{aligned}$$



## PART – C

1. The mean score of 1000 students in an examination is 36 and standard deviation is 16. If the score of the students is normally distributed how many students are expected to score more than 60 marks.

**Solution:**

Given: Mean =  $\mu = 36$

S.D =  $\sigma = 16$

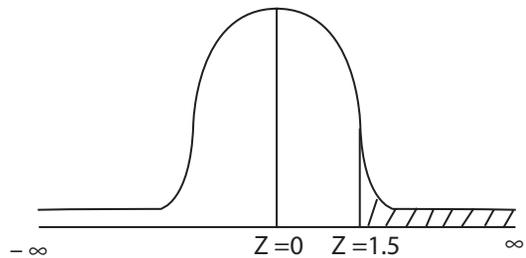
Number of students = 1000

We know  $Z = \frac{X - \mu}{\sigma}$

When  $X = 60$

$$\begin{aligned} Z &= \frac{60 - 36}{16} \\ &= \frac{24}{16} \\ &= \frac{3}{2} = 1.5 \end{aligned}$$

$$\therefore P(X > 60) = P(Z > 1.5)$$



$$\begin{aligned} P(X > 60) &= P(0 < Z < \infty) - P(0 < Z < 1.5) \\ &= 0.5 - 0.4332 = 0.0668 \end{aligned}$$

Probability for one student to score more than 60 = 0.0668

The number of students expected to score more than 60 marks out of 1000 students

$$= 0.0668 \times 1000$$

$$= 66.8$$

$$\cong 67$$

2. When  $X$  is normally distributed with mean 12, standard deviation is 4. Find (i)  $P(X \geq 20)$ , (ii)  $P(0 < X < 12)$  (iii)  $P(X \leq 20)$ .

**Solution:**

Given: Mean =  $\mu = 12$

S.D =  $\sigma = 4$

We know  $Z = \frac{X - \mu}{\sigma}$

$$\text{When } X = 20, Z = \frac{20 - 12}{4} = 2$$

$$\text{When } X = 0, Z = \frac{0 - 12}{4} = -3$$

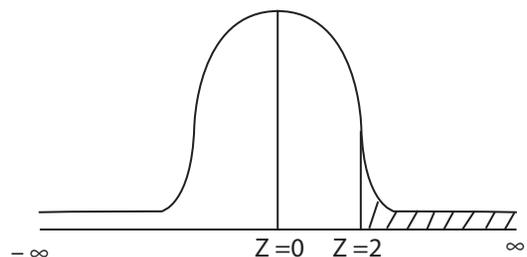
$$\text{When } X = 12, Z = \frac{12 - 12}{4} = 0$$

$$(i) P(X \geq 20) = P(Z \geq 2)$$

$$= P(0 \leq Z \leq \infty) - P(0 \leq Z \leq 2)$$

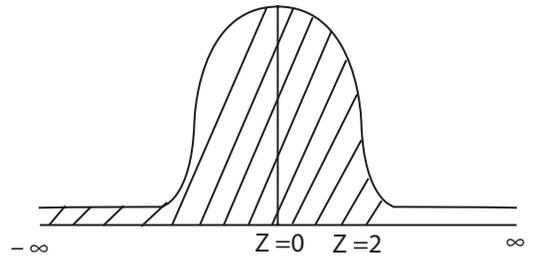
$$= 0.5 - 0.4772$$

$$= 0.0228$$



$$\begin{aligned}
 \text{(ii)} \quad P(0 < X < 12) &= P(-3 < Z < 0) \\
 &= P(0 < Z < 3) \\
 &= 0.4987
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad P(X \leq 20) &= P(Z \leq 2) \\
 &= P(-\infty < Z < 0) + P(0 < Z < 2) \\
 &= 0.5 + 0.4772 \\
 &= 0.9772
 \end{aligned}$$



3. In an aptitude test administered to 900 students, the scores obtained by the students are distributed normally with mean 50 and standard deviation 20. Find the number of students whose score is
- (i) between 30 and 70.

**Solution:**

Given: Mean =  $\mu = 50$

S.D =  $\sigma = 20$

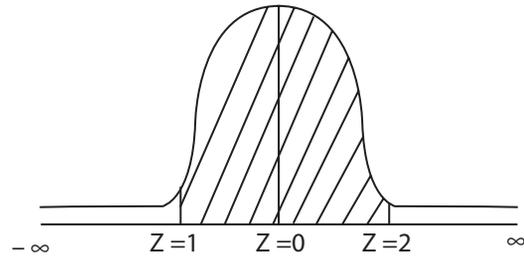
Number of students = 900

We know  $Z = \frac{X - \mu}{\sigma}$

When  $X = 30$ ;  $Z = \frac{30 - 50}{20} = -1$

When  $X = 70$ ;  $Z = \frac{70 - 50}{20} = 1$

$$\begin{aligned}
 \therefore P(30 < X < 70) &= P(-1 < Z < 1) \\
 &= P(-1 < Z < 0) + P(0 < Z < 1) \\
 &= P(0 < Z < 1) + P(0 < Z < 1) \\
 &= 2P(0 < Z < 1) \\
 &= 2 \times 0.3413 \\
 &= 0.6826
 \end{aligned}$$



Probability for a students score is between 30 and 70 = 0.6826

The number of students whose score is between 30 and 70 out of 900 students

$$\begin{aligned}
 &= 0.6826 \times 900 \\
 &= 614.34 \\
 &\cong 614
 \end{aligned}$$

4. Obtain  $K$ ,  $\mu$ ,  $\sigma^2$  of the normal distribution whose probability distribution function is  $f(x) = Ke^{-2x^2+4x}$ ,  $-\infty < x < \infty$ .

**Solution:**

The normal distribution is  $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ ,  $-\infty < x < \infty$ .

First consider, the power of the exponential (e)

$$\begin{aligned}
 \text{i.e.} \quad -2x^2 + 4x &= -2(x^2 - 2x) \\
 &= -2(x^2 - 2x + 1 - 1) \\
 &= -2[(x-1)^2 - 1] \\
 &= -2(x-1)^2 + 2 \\
 e^{-2x^2+4x} &= e^{-2(x-1)^2+2} \\
 &= e^2 \cdot e^{-\frac{(x-1)^2}{1/2}} \\
 &= e^2 \cdot e^{-\frac{1/2(x-1)^2}{(1/2)^2}} \\
 &= e^2 \cdot e^{-\frac{1}{2}\left[\frac{x-1}{1/2}\right]^2}
 \end{aligned}$$

Comparing with  $f(x)$

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = Ke^2 \cdot e^{-\frac{1}{2}\left[\frac{x-1}{1/2}\right]^2}$$

$$\text{We get } \sigma = \frac{1}{2}, \mu = 1, Ke^2 = \frac{1}{\sigma\sqrt{2\pi}}$$

$$\begin{aligned}
 \therefore K &= \frac{1}{\frac{1}{2}\sqrt{2\pi}} \cdot e^{-2} \\
 &= \frac{2e^{-2}}{\sqrt{2\pi}} \\
 &= \sqrt{\frac{2}{\pi}} \cdot e^{-2}
 \end{aligned}$$

## 2.3 CURVE FITTING

### Introduction:

Some of the methods of fitting curves are graphical method, which is a rough method and method of group average in which the evaluations of constants vary from one grouping to another grouping of data. So we adopt the method of least squares which gives a unique set of values to the constants in the equation of the fitting curve.

### Fitting a straight line by the method of least squares:

Let us consider the fitting of a straight line  $y = ax + b$  to the set of  $n$  points  $(x_i, y_i), i = 1, 2, \dots, n$  our aim is to determine  $a$  and  $b$  so that  $y = ax + b$  is the line of best fit. We apply the method of least squares to find the value of  $a$  and  $b$ .

The principle of least square consist in minimizing the sum of the square of the deviations of the actual values  $y$  from its estimated values as given by the line of best fit.

Let  $P(x_i, y_i), i = 1, 2, \dots, n$  be any general point in the Scatter diagram. Let  $y = f(x)$  be the relation suggested between  $x$  and  $y$ .

Let the ordinate at  $P_1$  meet  $y = f(x)$  at  $Q_1$  and  $x$ -axis at  $\mu_1$ .

$$\mu_1 Q_1 = f(x_1) \text{ \& } \mu_1 P_1 = y_1$$

$$Q_1 P_1 = \mu_1 P_1 - \mu_1 Q_1 = y_1 - f(x_1), (i = 1, 2, \dots, n)$$

$d_i = y_i - f(x_i)$  is called the residual at  $x = x_i$  some of the  $d_i$ 's may be possible and some may be negative.

$$E = \sum_{i=1}^n d_i^2 = \sum_{i=1}^n [y_i - f(x_i)]^2 \text{ is the sum of the square of the residual.}$$

If  $E = 0$ , i.e each  $d_i = 0$  then all the  $n$  points  $P_i$  will lie on  $Y = f(x)$ . If not, we will make  $f(x)$ , closer such that  $E$  is minimum this principle is known as the principle of least squares.

$$E = \sum_{i=1}^n [y_i - (ax_i + b)]^2$$

By the principle of least square,  $E$  is minimum

$$\therefore \frac{\partial E}{\partial a} = 0 \text{ and } \frac{\partial E}{\partial b} = 0$$

$$\text{i.e } \sum_{i=1}^n (x_i y_i - ax_i^2 - bx_i) = 0 \text{ and } \sum_{i=1}^n (y_i - ax_i - b) = 0$$

$$\text{i.e } a \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i = \sum_{i=1}^n x_i y_i \quad \dots\dots(1)$$

$$\& a \sum_{i=1}^n x_i + nb = \sum_{i=1}^n y_i \quad \dots\dots(2)$$

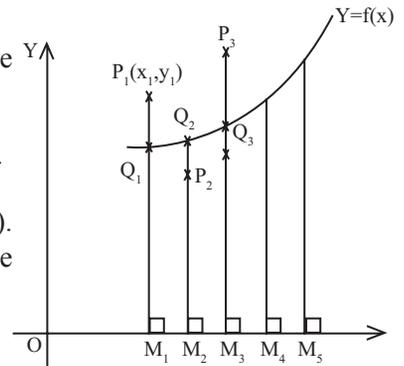
Solving equation (1) & (2) we get the values of  $a$  &  $b$ . To obtain the line of best fit.

### Note:

- (i) Equations (1) & (2) are called normal equations.
- (ii) Equations (1) & (2) can also be written as

$$a \sum x^2 + b \sum x = \sum xy$$

$$a \sum x + nb = \sum y \text{ by dropping suffix.}$$



**PART – C**

1. Fit a straight line to the data given below:

x	0	1	2	3	4
y	1	1.8	3.3	4.5	6.3

**Solution:**

Let the straight line be  $y = ax + b$ .

The normal equations are

$$a\sum x + nb = \sum y \quad \dots\dots (1)$$

$$a\sum x^2 + b\sum x = \sum xy \quad \dots\dots (2)$$

x	y	$x^2$	xy
0	1	0	0
1	1.8	1	1.8
2	3.3	4	6.6
3	4.5	9	13.5
4	6.3	16	25.2
<hr/>			
$\sum x=10$	$\sum y=16.9$	$\sum x^2=30$	$\sum xy=47.1$
			$n = 5$

Substituting the values, we get

$$10a + 5b = 16.9 \quad \dots\dots (3)$$

$$30a + 10b = 47.1 \quad \dots\dots (4)$$

Solving equations (3) & (4) we get

$$a = 1.33, b = 0.72$$

Hence the equation of best fit is  $y = 1.33x + 0.72$ .

2. Fit a straight line for the following data:

x	0	12	24	36	48
y	25	55	65	80	90

**Solution:**

$$\text{Let } u = \frac{X - 24}{12}, \quad v = \frac{Y - 65}{10}$$

Let the line in the new variable be  $v = au + b$ .

The normal equations are

$$a\sum u + nb = \sum v \quad \dots\dots(1)$$

$$a\sum u^2 + b\sum u = \sum uv \quad \dots\dots(2)$$

x	y	u	v	$u^2$	uv
0	35	-2	-3	4	6
12	55	-1	-1	1	1
24	65	0	0	0	0
36	80	1	1.5	1	1.5
48	90	2	2.5	4	5
<hr/>					
$\sum u = 0$	$\sum v = 0$	$\sum u^2 = 10$	$\sum uv = 13.5,$		$n = 5$

Substituting the values we get,

$$5b = 0 \Rightarrow b = 0$$

$$10a + 0 = 13.5$$

$$a = 1.35$$

The equation is  $V = 1.35u$

$$\left(\frac{y-65}{10}\right) = 1.35\left(\frac{x-24}{12}\right)$$

$$y - 65 = \frac{13.5}{12}(x - 24)$$

$$y = 1.125x - 27 + 65$$

$$y = 1.125x + 38$$

This is the equation of best fit.

3. Fit a straight line for the following data:

Year:	1984	1985	1986	1987	1988	1989
Production:	7	9	12	15	18	23

(in tones)

Estimate the production for the year 1990.

**Solution:**

Let  $x$  denotes the year and  $y$  denotes the production (in tones)

$$\text{Let } u = X - 1987$$

$$v = Y - 15$$

Let the line of best fit be  $v = au + b$

$\therefore$  The normal equations are

$$a\Sigma u + nb = \Sigma v \quad \dots\dots(1)$$

$$a\Sigma u^2 + b\Sigma u = \Sigma uv \quad \dots\dots(2)$$

x	y	u	v	u <sup>2</sup>	uv
1984	7	-3	-8	9	24
1985	9	-2	-6	4	12
1986	12	-1	-3	1	3
1987	15	0	0	0	0
1988	18	1	3	1	3
1989	23	2	8	4	16
		$\Sigma u = -3$	$\Sigma v = 6$	$\Sigma u^2 = 19$	$\Sigma uv = 58, n = 6$

Substituting the values

$$-3a + 6b = -6 \quad \dots\dots\dots(3)$$

$$19a - 3b = 58 \quad \dots\dots\dots(4)$$

Solving (3) & (4) we get

$$a = 3.142, \quad b = 0.571$$

$\therefore$  The equation of best fit is

$$v = au + b$$

$$(y - 15) = 3.142(X - 1987) + 0.571$$

The production for the year 1990 is

$$y - 15 = 3.142(1990 - 1987) + 0.571$$

$$\begin{aligned} y &= 3.142(3) + 0.571 + 15 \\ &= 24.997 \text{ tonnes} \end{aligned}$$

**EXERCISE****PART – A**

- For a poisson distribution if  $n = 1000$  and  $\lambda = 1$  find  $p$ .
- If  $P(X = 3) = P(x = 4)$  find the mean of the poisson distribution.
- The variance of a poisson distribution is 0.35. Find  $P(X = 2)$ .
- Give any two examples of poisson distribution.
- Define poisson distribution.
- Define normal distribution.
- Write any two properties of the normal curve.
- If  $X$  is normally distributed with mean 6 and S.D = 5. Express  $P(0 \leq X \leq 8)$  interms of standard normal distribution.
- Write the normal equations for the straight line  $y = ax + b$ .
- Let  $Z$  be a standard normal variate calculate
  - $P(-1.2 \leq Z \leq -0.5)$
  - $P(-2 \leq z \leq 3)$

**PART – B**

- Find the probability that no defective fuse will be found in a box of 200 fuses if experience show that 2% such fuses are defective.
- In a poisson distribution  $P(X = 1) = P(X = 2)$  find  $P(X = 0)$ .
- In a normal distribution mean is 12 and the standard deviation is 2. Find  $P(6 \leq X \leq 18)$ .
- Find  $\Sigma x$ ,  $\Sigma y$ ,  $\Sigma x^2$ ,  $\Sigma xy$  for the following data:

x	4	8	12	16	20	24
y	12	15	19	22	26	30

**PART – C**

- Let  $X$  be a poisson variate such that  $P(X = 1) = 0.2$  and  $P(X = 2) = 0.15$ . Find  $P(X = 0)$ .
- The probability that a man aged 50 years will die within a year is 0.01125. What is the probability that of 12 such men at least 11 will reach their fifty first birthday?
- The number of accidents in a year involving taxi drivers in a city follow a poisson distribution with mean equal to 3. Out of 1000 taxi drivers find approximately the number of driver with (i) no accident in a year (ii) more than 3 accident in a year.
- If  $X$  is normally distributed with mean 6 and standard deviation 5, find
  - $P(0 \leq X \leq 8)$
  - $P(1 \times -61 < 10)$
- The life of army shoes is normally distributed with mean 8 months and S.D 2 months. If 5000 pairs are issued, how many pairs would be expected to need replacement after 12 months.
- Find  $C$ ,  $\mu$  and  $\sigma^2$  of the normal distribution whose probability function is given by
 
$$f(x) = Ce^{-x^2+3x}, -\infty < x < \infty.$$
- Students of a class were given a aptitude test. This marks were found to be normally distributed with mean 60 and standard deviation 5. What percent of students scored (i) more than 60 marks (ii) between 45 and 65 marks.

8. Fit a straight line to the following data:

x	2	4	6	8	10	12
y	7	10	12	14	17	24

9. Fit a straight line to the following data by the method of least squares.

x	5	10	15	20	25
y	15	19	23	26	30

10. The following table shows the number of students in a post graduate course.

Year	1922	1993	1994	1995	1996
No. of Students	28	38	46	40	56

### ANSWERS

#### PART – A

1. 0.001      2. 4      3.  $\frac{e^{-0.35}(0.35)^2}{2}$       8.  $P(-1.2 < Z < 0.4)$       10. (i) 0.8787      (ii) 0.9759

#### PART – B

1.  $e^{-4}$       2.  $e^{-2}$       3. 0.9974      4.  $\Sigma x = 84, \Sigma y = 124, \Sigma x^2 = 1456, \Sigma xy = 1988$

#### PART – C

1.  $e^{-1.5}$       2. 0.9916      3. (i) 50 (approx.)      (ii) 353 (approx.)      4. (i) 0.5403      (ii) 0.9544

5. 4886 shoes      6.  $\mu = \frac{3}{2}, \sigma = \frac{1}{\sqrt{2}}, c = \frac{e^{9/4}}{\sqrt{\pi}}$       7. (i) 50%      (ii) 84%

8.  $Y = 1.542x + 26.794$       9.  $Y = 0.7x + 12.3$       10.  $y_{1997} = 59$

## UNIT – III

# APPLICATION OF DIFFERENTIATION

3.1 VELOCITY AND ACCELERATION : Velocity and Acceleration – Simple Problems.

3.2 TANGENT AND NORMAL: Tangent and Normal – Simple Problems.

3.3 MAXIMA AND MINIMA

Definition of increasing and decreasing functions and turning points. Maxima and Minima of single variable only – Simple Problems.

## 3.1 VELOCITY AND ACCELERATION

### Definition: Velocity

The rate of change of displacement with respect to time is defined as velocity. If 's' denotes the displacement, 't' denotes the time then the velocity is  $v = \frac{ds}{dt}$ .

### Definition: Acceleration

The rate of change of velocity with respect to time is defined as acceleration and it is denoted by 'a'.

The acceleration,  $a = \frac{dv}{dt} = \frac{d}{dt} \left( \frac{ds}{dt} \right) = \frac{d^2s}{dt^2}$ .

**Note : 1**

- (i) At time  $t = 0$ , the velocity is initial velocity.
- (ii) When the particle comes to rest, then velocity  $v = 0$ .
- (iii) When the acceleration is zero, velocity is uniform.

**Note : 2**

If  $s = f(t)$  is the given distance time relation then its first derivative with respect to 't' is velocity and the second derivative with respect to 't' is acceleration.

## WORKED EXAMPLES

### PART – A

1. The distance time formula of a moving particle is  $s = 2 \cos 3t$ . Find its velocity and acceleration.

**Solution:**

Given:  $s = 2 \cos 3t$

Velocity:  $v = \frac{ds}{dt} = 2[-3 \sin 3t] = -6 \sin 3t$

Acceleration:  $a = \frac{d^2s}{dt^2} = -6[3 \cos 3t] = -18 \cos 3t$

2. If  $s = 5t^3 - 3t^2 + 10t$ . Find the velocity and acceleration after 't' seconds.

**Solution:**

Given:  $s = 5t^3 - 3t^2 + 10t$

Velocity:  $v = \frac{ds}{dt} = 5[3t^2] - 3[2t] + 10[1]$

$$\Rightarrow v = 15t^2 - 6t + 10$$

Acceleration:  $a = \frac{d^2s}{dt^2} = 15[2t] - 6[1] + 0$

$$\Rightarrow a = 30t - 6$$

3. If  $s = 3t^2 - 5t + 7$  find initial velocity.

**Solution:**

Given:  $s = 3t^2 - 5t + 7$

$\therefore$  Velocity:  $v = \frac{ds}{dt} = 3[2t] - 5[1] + 0$

$$\Rightarrow v = 6t - 5$$

$\therefore$  Initial Velocity =  $\left(\frac{ds}{dt}\right)_{t=0}$   
 $= 6(0) - 5 = -5$

4. The distance travelled by a particle in time 't' seconds is given by  $s = t^2 - 12t + 3$ . Find the time 't' when the velocity becomes zero.

**Solution:**

Given:  $s = t^2 - 12t + 3$

Velocity:  $v = \frac{ds}{dt} = [2t] - 12[1] + 0$

$$\Rightarrow v = 2t - 12$$

When velocity becomes zero then

$$v = 0 \Rightarrow 2t - 12 = 0$$

$$2t = 12$$

$$\Rightarrow \boxed{t = 6 \text{ sec}}$$

when  $t = 6$  seconds, velocity becomes zero.

5. The distance time formula of a moving particle is given by  $s = 7t^3 - 5t^2 + 12t - 13$ . Find the initial acceleration.

**Solution:**

Given:  $s = 7t^3 - 5t^2 + 12t - 13$

Velocity:  $v = \frac{ds}{dt} = 7[3t^2] - 5[2t] + 12[1] - 0$

$$\Rightarrow \boxed{v = 21t^2 - 10t + 12}$$

Acceleration:  $a = \frac{d^2s}{dt^2} = 21[2t] - 10[1] + 0$

$$\Rightarrow \boxed{a = 42t - 10}$$

$\therefore$  Initial Acceleration =  $\left(\frac{d^2s}{dt^2}\right)_{t=0}$   
 $= 42(0) - 10 = -10 \text{ units / sec}^2$

6. If  $s = ae^t + be^{-t}$ , show that the acceleration is always equal to its distance.

**Solution:**

$$\text{Given: } s = ae^t + be^{-t} \text{ ————— (1)}$$

$$\text{Velocity: } v = \frac{ds}{dt} = a(e^t) + b(-e^{-t})$$

$$\Rightarrow v = ae^t - be^{-t} \text{ .....(2)}$$

$$\text{Acceleration: } a = \frac{d^2s}{dt^2} = a[e^t] - b[-e^{-t}]$$

$$\Rightarrow a = ae^t + be^{-t}$$

$$\Rightarrow \boxed{a = s} \quad \langle \because \text{ of (1)} \rangle$$

$\therefore$  The acceleration is always equal to its distance.

### PART – B

1. If  $s = 2t^3 - 3t^2 + 1$  find the velocity and acceleration after  $t = 2$  sec.

**Solution:**

$$\text{Given: } s = 2t^3 - 3t^2 + 1$$

$$\text{Velocity: } v = \frac{ds}{dt} = 2(3t^2) - 3(2t) + 0$$

$$\Rightarrow \boxed{v = 6t^2 - 6t}$$

$$\text{Acceleration: } a = \frac{d^2s}{dt^2} = 6[2t] - 6[1]$$

$$\Rightarrow \boxed{a = 12t - 6}$$

After  $t = 2$  secs:

$$\begin{aligned} \text{(i) Velocity: } v &= 6(2)^2 - 6(2) \\ &= 6(4) - 12 \\ &= 24 - 12 \end{aligned}$$

$$\Rightarrow \boxed{v = 12 \text{ unit / sec}}$$

$$\begin{aligned} \text{(ii) Acceleration: } a &= 12(2) - 6 \\ &= 24 - 6 \end{aligned}$$

$$\Rightarrow \boxed{a = 18 \text{ unit / sec}^2}$$

2. The distance travelled by a particle at time 't' sec is given by  $s = ae^{2t} + be^{-2t}$ . Show that the acceleration is always equal to four times the distance travelled.

**Solution:**

$$\text{Given: } \boxed{s = ae^{2t} + be^{-2t}} \text{ .....(1)}$$

$$\text{Velocity: } v = \frac{ds}{dt} = a(2e^{2t}) + b(-2e^{-2t})$$

$$\Rightarrow \boxed{v = 2ae^{2t} - 2be^{-2t}} \text{ .....(2)}$$

$$\text{Acceleration: } a = \frac{d^2s}{dt^2} = 2a(2e^{2t}) - 2b(-2e^{-2t})$$

$$= 4ae^{2t} + 4be^{-2t}$$

$$= 4(ae^{2t} + be^{-2t})$$

$$\Rightarrow \boxed{a = 4s} \quad \langle \because \text{ of (1)} \rangle$$

$\therefore$  The acceleration is always equal to four times the distance travelled.

3. The velocity 'v' of a particle moving along a straight line when at a distance 's' from the origin is expressed by  $s^2 = m + nv^2$ , show that the acceleration of the particle is  $\frac{s}{n}$ .

**Solution:**

$$\text{Given: } s^2 = m + nv^2$$

Differentiate w.r. to 't' on both sides

$$2s \frac{ds}{dt} = 0 + n \cdot 2v \frac{dv}{dt}$$

$$2sv = 2nv \cdot a$$

$$s = n \cdot a$$

$$\Rightarrow a = \frac{s}{n}$$

$$\Rightarrow \text{acceleration : } a = \frac{s}{n}$$

$$\left[ \begin{array}{l} \text{We have} \\ \frac{ds}{dt} = v \\ \frac{dv}{dt} = a \end{array} \right.$$

### PART – C

1. The distance travelled by a particle along a straight line is given by  $s = 2t^3 + 3t^2 - 72t + 1$ . Find the acceleration when the velocity vanishes and find initial velocity.

**Solution:**

$$\text{Given: } \boxed{s = 2t^3 + 3t^2 - 72t + 1} \quad \dots\dots\dots(1)$$

$$\text{Velocity : } v = \frac{ds}{dt} = 2(3t^2) + 3(2t) - 72(1) + 0$$

$$\Rightarrow \boxed{v = 6t^2 + 6t - 72} \quad \dots\dots\dots(2)$$

$$\text{Acceleration : } a = \frac{d^2s}{dt^2} = 6(2t) + 6(1) - 0$$

$$\Rightarrow \boxed{a = 12t + 6} \quad \dots\dots\dots(3)$$

(i) To find acceleration when the velocity vanishes

$$\text{When } v = 0 \Rightarrow 6t^2 + 6t - 72 = 0$$

$$\div 6 ; t^2 + t - 12 = 0$$

$$(t + 4)(t - 3) = 0$$

$$t = -4, t = 3$$

Here  $t = -4$  is not possible  $\Rightarrow t = 3$  sec.

$\therefore$  When  $t = 3$  sec.

$$\text{Acceleration, } a = 12(3) + 6 \quad \text{(using (3))}$$

$$a = 36 + 6$$

$$\Rightarrow a = 42 \text{ unit/sec}^2$$

Acceleration is 42 unit/sec<sup>2</sup> when the velocity vanish.

(ii) To find initial velocity:

Put  $t = 0$  in (2) then

$$\text{Velocity : } v = 6(0) + 6(0) - 72.$$

$$\Rightarrow \boxed{v = -72 \text{ unit / sec}}$$

Initial velocity is  $-72$  unit/sec.

2. The distance 's' metres at time 't' seconds travelled by a particle is given by  $s = t^3 - 9t^2 + 24t - 18$ . Find the velocity when acceleration is zero and also find the acceleration when the velocity is zero.

**Solution:**

$$\text{Given: } \boxed{s = t^3 - 9t^2 + 24t - 18} \quad \dots\dots\dots(1)$$

$$\text{Velocity: } v = \frac{ds}{dt} = (3t^2) - 9(2t) + 24(1) - 0$$

$$\Rightarrow \boxed{v = 3t^2 - 18t + 24} \quad \dots\dots\dots(2)$$

$$\text{Acceleration: } a = \frac{d^2s}{dt^2} = 3(2t) - 18(1) + 0$$

$$\Rightarrow \boxed{a = 6t - 18} \quad \dots\dots\dots(3)$$

- (i) To find velocity when the acceleration is zero

$$\text{When } a = 0 \Rightarrow 6t - 18 = 0$$

$$6t = 18$$

$$\Rightarrow \boxed{t = 3 \text{ sec}}$$

When  $t = 3$  secs

$$\begin{aligned} \text{Velocity: } v &= 3(3)^2 - 18(3) + 24 \\ &= 27 - 54 + 24 \end{aligned}$$

$$\Rightarrow \boxed{v = -3 \text{ m/sec}}$$

Velocity is  $-3$  m/sec when acceleration is zero.

- (ii) To find acceleration when the velocity is zero.

$$\text{When } v = 0 \Rightarrow 3t^2 - 18t + 24 = 0$$

$$\div 3 \quad ; \quad t^2 - 6t + 8 = 0$$

$$(t - 2)(t - 4) = 0$$

$$t = 2, t = 4$$

When  $t = 2$  secs

$$\begin{aligned} \text{Acceleration: } a &= 6(2) - 18 \\ &= 12 - 18 \end{aligned}$$

$$\Rightarrow \boxed{a = -6 \text{ m/sec}^2}$$

When  $t = 4$  secs

$$\begin{aligned} \text{Acceleration, } a &= 6(4) - 18 \\ &= 24 - 18 \end{aligned}$$

$$\Rightarrow \boxed{a = 6 \text{ m/sec}^2}$$

Acceleration is  $6$  and  $-6$  m/sec<sup>2</sup> when velocity is zero.

3. If the distance 'x' of a particle in recti linear motion in 't' secs is given by  $x = a \cos\left(\frac{\pi}{2}t\right) + b \sin\left(\frac{\pi}{2}t\right)$   
Prove that the acceleration varies as its displacement.

**Solution:**

$$\text{Given: } \boxed{x = a \cos\left(\frac{\pi}{2}t\right) + b \sin\left(\frac{\pi}{2}t\right)} \dots\dots\dots(1)$$

$$\text{Velocity: } v = \frac{dx}{dt} = a \left[ -\frac{\pi}{2} \sin\left(\frac{\pi}{2}t\right) \right] + b \left[ \frac{\pi}{2} \cos\left(\frac{\pi}{2}t\right) \right]$$

$$\Rightarrow \boxed{v = -\frac{\pi}{2}a \sin\left(\frac{\pi}{2}t\right) + \frac{\pi}{2}b \cos\left(\frac{\pi}{2}t\right)} \dots\dots\dots(3)$$

$$\begin{aligned} \text{Acceleration: } a &= \frac{d^2x}{dt^2} = -\frac{\pi}{2}a \left[ \frac{\pi}{2} \cos\left(\frac{\pi}{2}t\right) \right] + \frac{\pi}{2}b \left( -\frac{\pi}{2} \sin\left(\frac{\pi}{2}t\right) \right) \\ &= -\left(\frac{\pi}{2}\right)^2 a \cos\left(\frac{\pi}{2}t\right) - \left(\frac{\pi}{2}\right)^2 b \sin\left(\frac{\pi}{2}t\right) \\ &= -\left(\frac{\pi}{2}\right)^2 \left[ a \cos\left(\frac{\pi}{2}t\right) + b \sin\left(\frac{\pi}{2}t\right) \right] \\ &= -\left(\frac{\pi}{2}\right)^2 [x] \quad \langle \text{using (1)} \rangle \\ &\Rightarrow \boxed{a \propto x} \end{aligned}$$

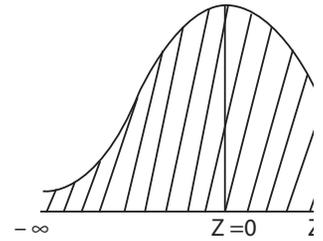
$\therefore$  The acceleration varies as its displacement.

## 3.2 TANGENT AND NORMAL

### Definition: Tangent

Let P be any point on the curve  $y = f(x)$  and let Q be any other point on the curve close to P. The limiting position of the PQ as Q approaches to P is called the tangent to the curve  $y = f(x)$  at the point P.

The tangent is a straight line which touches the curve at exactly one point.

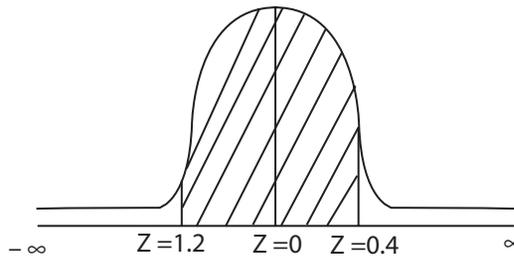


### Definition: Normal

The normal is a straight line passing through the point P and perpendicular to the tangent PT.

### Geometrical meaning of $\frac{dy}{dx}$ :

Let P ( $x, y$ ) and Q ( $x + \Delta x, y + \Delta y$ ) be the two neighbouring points on the curve  $y = f(x)$ . Let PQ makes an angle  $\theta$  with x-axis and the tangent PT at P makes an angle  $\psi$  with x-axis.



Now, slope of the chord. PQ:

$$\tan \theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y + \Delta y - y}{x + \Delta x - x} = \frac{\Delta y}{\Delta x}$$

$$\therefore \text{Slope of PQ : } \tan \theta = \frac{\Delta y}{\Delta x}$$

As  $Q \rightarrow P$  ;  $\Delta x \rightarrow 0$  and  $\theta \rightarrow \psi$  and hence chord becomes the tangent.

$\therefore$  Slope of line tangent at P is,

$$\lim_{\theta \rightarrow \psi} \tan \theta = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \quad \tan \psi = \frac{dy}{dx}$$

Hence for the curve  $y = f(x)$ , the slope of tangent is  $\frac{dy}{dx}$

**Results:**

1. Slope of the tangent to the curve  $y = f(x)$  at  $(x_1, y_1)$  is  $m = \left( \frac{dy}{dx} \right)_{(x_1, y_1)}$ .
2. Slope of normal  $= -\frac{1}{m}$ .
3. The equation of tangent to the curve  $y = f(x)$  at  $(x_1, y_1)$  is  $y - y_1 = m(x - x_1)$ .
4. The equation of normal to the curve  $y = f(x)$  at  $(x_1, y_1)$  is  $y - y_1 = -\frac{1}{m}(x - x_1)$ .

**WORKED EXAMPLES****PART – A**

1. Find the slope of the tangent to the curve  $y = x^3$  at the point where  $x = \frac{1}{2}$ .

**Solution:**

Given:  $y = x^3$

$$\Rightarrow \frac{dy}{dx} = 3x^2$$

$$\begin{aligned} \therefore \text{Slope of tangent : } m &= \left( \frac{dy}{dx} \right)_{x=\frac{1}{2}} \\ &= 3 \left( \frac{1}{2} \right)^2 = 3 \left( \frac{1}{4} \right) \end{aligned}$$

$$\Rightarrow \boxed{m = \frac{3}{4}}$$

2. Find the slope of the tangent to the curve  $y^2 = 4ax$  at  $(at^2, 2at)$ .

**Solution:**

Given:  $y^2 = 4ax$

$$2y \frac{dy}{dx} = 4a(1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore \text{Slope of tangent : } m = \left( \frac{dy}{dx} \right)_{(at^2, 2at)}$$

$$= \frac{2a}{2at}$$

$$\Rightarrow \boxed{m = \frac{1}{t}}$$

3. Find the slope of the tangent at  $(1, -2)$  on the curve  $y = x^4 - 3x^2$ .

**Solution:**

$$\text{Given: } y = x^4 - 3x^2$$

$$\frac{dy}{dx} = [4x^3] - 3[2x]$$

$$\Rightarrow \frac{dy}{dx} = 4x^3 - 6x$$

$$\begin{aligned} \therefore \text{Slope of tangent : } m &= \left( \frac{dy}{dx} \right)_{(1,-2)} \\ &= 4(1)^3 - 6(1) \\ &= 4 - 6 \\ &\Rightarrow \boxed{m = -2} \end{aligned}$$

4. Find the slope of normal to the curve  $y = x^2 + 7x$  at  $(1, 8)$ .

**Solution:**

$$\text{Given: } y = x^2 + 7x$$

$$\frac{dy}{dx} = (2x) + 7(1)$$

$$\Rightarrow \frac{dy}{dx} = 2x + 7$$

$$\begin{aligned} \therefore \text{Slope of tangent : } m &= \left( \frac{dy}{dx} \right)_{(1,8)} \\ &= 2(1) + 7 = 2 + 7 \\ &\Rightarrow \boxed{m = 9} \end{aligned}$$

$$\therefore \text{Slope normal } := \frac{1}{m} = \frac{-1}{9}$$

5. At what point does the curve  $xy = x - 4$  cuts the x-axis.

**Solution:**

$$\text{Given: } xy = x - 4 \quad \dots\dots\dots (1)$$

When the curve cuts the x-axis then put  $y = 0$  in (1) we get,

$$x(0) = x - 4 \Rightarrow x - 4 = 0$$

$$\Rightarrow x = 4$$

$\therefore$  The required point is  $(4, 0)$ .

## PART – B

1. Find the equation of tangent at the point  $(1, 1)$  on the curve  $x^2 + 2y^2 = 3$ .

**Solution:**

$$\text{Given: } x^2 + 2y^2 = 3$$

Differentiate w.r. to 'x' on both sides

$$2x + 2 \left[ 2y \frac{dy}{dx} \right] = 0$$

$$\div 2; x + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{x}{2y}}$$

$$\therefore \text{Slope of tangent : } m = \left( \frac{dy}{dx} \right)_{(1,1)}$$

$$= \frac{-1}{2(1)}$$

$$\Rightarrow \boxed{m = -\frac{1}{2}}$$

The equation of tangent is

$$y - y_1 = m(x - x_1) \quad \left| \quad (x_1, y_1) = (1, 1) \right.$$

$$y - 1 = -\frac{1}{2}(x - 1) \quad \left| \quad m = -\frac{1}{2} \right.$$

$$2y - 2 = -x + 1$$

$$\boxed{x + 2y - 3 = 0}$$

2. Find the equation of normal to the curve  $y = 4x - 3x^2$  at  $(2, -4)$ .

**Solution:**

$$\text{Given : } y = 4x - 3x^2$$

$$\frac{dy}{dx} = 4(1) - 3(2x) = 4 - 6x$$

$$\text{Slope of the tangent } m = \left( \frac{dy}{dx} \right)_{(2,-4)} = 4 - 6(2) = 4 - 12$$

$$\Rightarrow \boxed{m = -8}$$

$\therefore$  The equation of normal is

$$y - y_1 = -\frac{1}{m}(x - x_1) \quad \left| \quad (x_1, y_1) = (2, -4) \right.$$

$$y + 4 = \frac{1}{8}(x - 2) \quad \left| \quad m = -8 \right.$$

$$8y + 32 = x - 2$$

$$\Rightarrow \boxed{x - 8y - 34 = 0}$$

## PART – C

1. Find the equation of the tangent and normal to the curve  $y = 6 + x - x^2$  at  $(2, 4)$ .

**Solution:**

$$\text{Given: } y = 6 + x - x^2$$

$$\frac{dy}{dx} = 0 + (1) - (2x)$$

$$\Rightarrow \frac{dy}{dx} = 1 - 2x$$

To find slope:

$$\begin{aligned}\text{Slope : } m &= \left( \frac{dy}{dx} \right)_{(2,4)} \\ &= 1 - 2(2) = 1 - 4 = -3\end{aligned}$$

To find equation of tangent:

$$\text{Point: } (x_1, y_1) = (2, 4) \text{ \& } m = -3$$

∴ The equation of tangent is

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -3(x - 2)$$

$$y - 4 = -3x + 6$$

$$\boxed{3x + y - 10 = 0}$$

To find equation of normal:

$$\text{Point } (x_1, y_1) = (2, 4) \text{ \& } m = -3$$

∴ The equation of normal is,

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - 4 = \frac{1}{3}(x - 2)$$

$$3y - 12 = x - 2$$

$$\boxed{x - 3y + 10 = 0}$$

2. Find the equation of tangent and normal to the curve  $y = \frac{x+3}{x^2+1}$  at (2, 1).

**Solution:**

$$\begin{aligned}\text{Given : } y &= \frac{x+3}{x^2+1} \\ \frac{dy}{dx} &= \frac{(x^2+1)(1) - (x+3)(2x)}{(x^2+1)^2}\end{aligned}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+1-2x^2-6x}{(x^2+1)^2}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{1-6x-x^2}{(x^2+1)^2}}$$

To find Slope:

$$\begin{aligned}\text{Slope, } m &= \left( \frac{dy}{dx} \right)_{(2,1)} \\ &= \frac{1-6(2)-(2)^2}{[(2)^2+1]^2} = \frac{1-12-4}{(5)^2} = \frac{-15}{25} \\ &\Rightarrow \boxed{m = -\frac{3}{5}}\end{aligned}$$

To find equation of tangent:

$$\text{Point: } (x_1, y_1) = (2, 1) \text{ \& } m = -\frac{3}{5}$$

The equation of tangent is,

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -\frac{3}{5}(x - 2)$$

$$5y - 5 = -3x + 6$$

$$\boxed{3x + 5y - 11 = 0}$$

To find equation of normal:

Point  $(x_1, y_1) = (2, 1)$  &  $m = -\frac{3}{5}$

The equation of normal is,

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - 1 = \frac{5}{3}(x - 2)$$

$$3y - 3 = 5x - 10$$

$$\boxed{5x - 3y - 7 = 0}$$

3. Find the equation of the tangent and normal to the parabola  $y^2 = 4ax$  at  $(at^2, 2at)$ .

**Solution:**

Given:  $y^2 = 4ax$

Differentiate w.r. to 'x'

$$2y \frac{dy}{dx} = 4a(1)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2a}{y}}$$

To find slope:

$$\text{Slope, } m = \left( \frac{dy}{dx} \right)_{(at^2, 2at)} = \frac{2a}{y} = \frac{2a}{2at}$$

$$\Rightarrow \boxed{m = \frac{1}{t}}$$

To find equation of tangent:

Point :  $(x_1, y_1) = (at^2, 2at)$  &  $m = \frac{1}{t}$

The equation of tangent is,

$$y - y_1 = m(x - x_1)$$

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$yt - 2at^2 = x - at^2$$

$$x - yt + at^2 = 0$$

To find equation of normal:

$$\text{Point: } (x_1, y_1) = (at^2, 2at) \text{ \& } m = \frac{1}{t}$$

The equation of normal is,

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - 2at = -t(x - at^2)$$

$$y - 2at = -xt + at^3$$

$$\boxed{xt + y - 2at - at^3 = 0}$$

4. Find the equation of tangent to the curve  $y = (x - 3)(x - 4)$  at the point where the curve cuts the x-axis.

**Solution:**

$$\text{Given: } y = (x - 3)(x - 4) \quad \dots\dots\dots(1)$$

On the x-axis,  $y = 0$

$\therefore$  (1) becomes

$$(x - 3)(x - 4) = 0$$

$$x - 3 = 0 \quad | \quad x - 4 = 0$$

$$\boxed{x = 3} \quad | \quad \boxed{x = 4}$$

$\therefore$  The points are (3, 0) & (4, 0).

To find  $\frac{dy}{dx}$ :

$$\text{Since } y = (x - 3)(x - 4)$$

$$\frac{dy}{dx} = (x - 3)(1) + (x - 4)(1)$$

$$= x - 3 + x - 4$$

$$\frac{dy}{dx} = 2x - 7$$

Case (i) : When the point is (3, 0)

$$\text{Slope } m = \left( \frac{dy}{dx} \right)_{(3,0)}$$

$$= 2(3) - 7$$

$$= 6 - 7$$

$$\Rightarrow \boxed{m = -1}$$

To find equation of tangent:

$$\text{Point: } (x_1, y_1) = (3, 0) \text{ \& } m = -1$$

$\therefore$  The equation of tangent is,

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -1(x - 3)$$

$$y = -x + 3$$

$$\boxed{x + y - 3 = 0} =$$

Case (ii): When the point is (4, 0)

$$\begin{aligned}\text{Slope, } m &= \left( \frac{dy}{dx} \right)_{(4,0)} \\ &= 2(4) - 7 \\ &= 8 - 7 \\ &\Rightarrow \boxed{m = 1}\end{aligned}$$

To find equation of tangent

Point  $(x_1, y_1) = (4, 0)$  &  $m = 1$

∴ The equation of tangent is,

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 1(x - 4)$$

$$y = x - 4$$

$$\boxed{x - y - 4 = 0}$$

5. Find the equation of the tangent to the curve  $y = 3x^2 + 2x + 5$  at the point where the curve cuts the y-axis.

**Solution:**

$$\text{Given: } y = 3x^2 + 2x + 5 \quad \dots\dots\dots (1)$$

By data, the curve cuts the y-axis then  $x = 0$ .

$$\therefore (1) \text{ becomes, } y = 3(0) + 2(0) + 5$$

$$\Rightarrow y = 5$$

∴ The point is (0, 5)

$$\text{Now, } y = 3x^2 + 2x + 5$$

$$\frac{dy}{dx} = 3(2x) + 2(1) + 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 6x + 2}$$

To find Slope:

$$\text{Slope, } m = \left( \frac{dy}{dx} \right)_{(0,5)}$$

$$= 6(0) + 2$$

$$\Rightarrow \boxed{m = 2}$$

To find equation of tangent:

Point :  $(x_1, y_1) = (0, 5)$  &  $m = 2$

The equation of tangent is,

$$y - y_1 = m(x - x_1)$$

$$y - 5 = 2(x - 0)$$

$$y - 5 = 2x$$

$$\boxed{2x - y + 5 = 0}$$

6. Find the equation of the tangent to the curve  $y = 3x^2 - 6x + 1$  at the point where the tangent is parallel to the line  $6x - y + 2 = 0$ .

**Solution:**

$$\text{Given: } y = 3x^2 - 6x + 1 \quad \dots\dots\dots (1)$$

$$\frac{dy}{dx} = 3(2x) - 6(1) + 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 6x - 6}$$

Given equation of line is  $6x - y + 2 = 0$

$$\therefore \text{ Slope of the line} = -\frac{a}{b} = \frac{-6}{-1} = 6$$

By data, the tangent is parallel to the given line, then its slope is also  $m = 6$

$$\text{Thus, } \frac{dy}{dx} = m \Rightarrow 6x - 6 = 6 \Rightarrow x - 1 = 1 \Rightarrow \boxed{x = 2}$$

Put  $x = 2$  in (1) we get,

$$y = 3(2)^2 - 6(2) + 1$$

$$= 12 - 12 + 1$$

$$\Rightarrow y = 1$$

$\therefore$  The point is  $(2, 1)$ .

To find equation of tangent:

Point  $(x_1, y_1) = (2, 1)$  &  $m = 6$

$\therefore$  The equation of tangent is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 6(x - 2)$$

$$y - 1 = 6x - 12$$

$$\boxed{6x - y - 11 = 0}$$

### 3.3 MAXIMA AND MINIMA

**Increasing function:**

A function  $y = f(x)$  is said to be increasing function if value of  $y$  increases as  $x$  increases (or) value of  $y$  decreases as  $x$  decreases.

From the fig,  $y = f(x)$  is a increasing function and  $\Delta x > 0, \Delta y > 0$ .

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{+}{+} = \text{Positive}$$

Hence, at every point on the increasing function the value of  $\frac{dy}{dx}$  is positive.

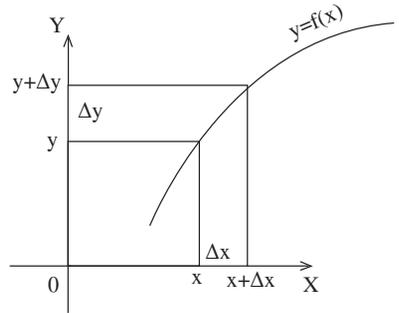
**Decreasing function:**

A function  $y = f(x)$  is said to be decreasing function if the value of  $y$  decreases as  $x$  increases (or) value of  $y$  increases as  $x$  decreases.

From the fig,  $y = f(x)$  is a decreasing function, also  $\Delta x > 0$  and  $\Delta y < 0$ .

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{-}{+} = \text{Negative}$$

Hence, at every point on the decreasing function the value of  $\frac{dy}{dx}$  is negative.



**Turning Points:**

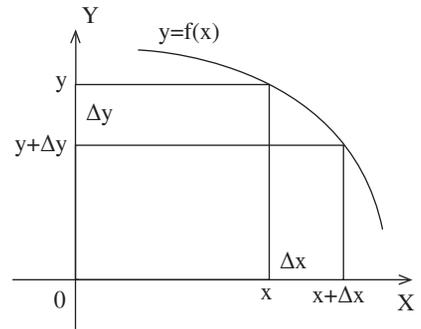
The points at which the function changes either from decreasing to increasing (or) increasing to decreasing are called turning points.

At the turning points the value of  $\frac{dy}{dx} = 0$ .

**Definition: Maximum**

Let  $y = f(x)$  be a continuous function in 'x'. The function  $y = f(x)$  increases in the interval (M, N) upto  $x = a$  and then decreases then the point  $x = a$  is called a maximum point and  $f(a)$  the value of the function at  $x = a$  is called the maximum value.

In the neighbourhood of  $x = a$ , the value of  $f(a)$  is greater than any other value of  $f(x)$ .



**Definition: Minimum**

The function  $y = f(x)$  decreases in the interval (P, Q) upto  $x = b$  and then increases then the point  $x = b$  is called minimum point and  $f(b)$  the value of the function at  $x = b$  is called the minimum value.

In the neighbourhood of  $x = b$ , the value of  $f(b)$  is smaller than any other value of  $f(x)$ .

**The condition for the function  $y = f(x)$  to be maximum at  $x = a$ :**

(i)  $\frac{dy}{dx} = 0$       (ii)  $\frac{d^2y}{dx^2} < 0$  (negative)

**The condition for the function  $y = f(x)$  to be minimum at  $x = b$**

(i)  $\frac{dy}{dx} = 0$       (ii)  $\frac{d^2y}{dx^2} > 0$  (positive)

**WORKED EXAMPLES**

**PART – A**

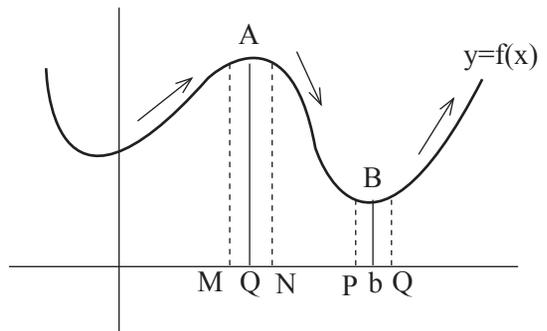
1. Show that  $e^x$  is an increasing function on R.

**Solution:**

Let  $y = e^x$

$$\frac{dy}{dx} = e^x > 0, x \in \mathbb{R}$$

$\therefore e^x$  is an increasing function on R.



2. Show that the function  $y = e^{-x}$  is decreasing function on  $[0, 1]$ .

**Solution:**

$$\text{Given: } y = e^{-x}$$

$$\frac{dy}{dx} = -e^{-x} < 0$$

$\therefore y = e^{-x}$  is a decreasing function on  $[0, 1]$ .

3. For what value of 'x' the function  $y = x^2 - 4x$  will have maximum (or) minimum value.

**Solution:**

$$\text{Given: } x^2 - 4x$$

$$\frac{dy}{dx} = 2x - 4$$

$$\text{If } \frac{dy}{dx} = 0 \Rightarrow 2x - 4 = 0 \Rightarrow x = 2$$

$\therefore$  At  $x = 2$ , the function will have maximum (or) minimum value.

4. Show that the function  $y = 4x - x^2 + 7$  is maximum at  $x = 2$ .

**Solution:**

$$\text{Given: } y = 4x - x^2 + 7$$

$$\frac{dy}{dx} = 4(1) - (2x) + 0 = 4 - 2x$$

$$\frac{d^2y}{dx^2} = 0 - 2(1) = -2$$

$$\text{If } \frac{dy}{dx} = 0 \Rightarrow 4 - 2x = 0 \Rightarrow x = 2$$

$$\therefore \text{ when } x = 2, \frac{d^2y}{dx^2} = -2 < 0$$

$\therefore$  The function is maximum at  $x = 2$ .

## PART – B

1. Find the maximum value of  $y = 4x - 2x^2$ .

**Solution:**

$$\text{Given : } y = 4x - 2x^2$$

$$\frac{dy}{dx} = 4(1) - 2(2x) = 4 - 4x$$

$$\frac{d^2y}{dx^2} = 0 - 4(1) = -4$$

$$\text{For if } \frac{dy}{dx} = 0 \Rightarrow 4 - 4x = 0 \Rightarrow \boxed{x = 1}$$

$$\text{When } x = 1 \Rightarrow \frac{d^2y}{dx^2} = -4 < 0 (-ve)$$

$\Rightarrow x = 1$  is a maximum point.

Put  $x = b$ , in (1) we get,

$$\begin{aligned}
 y &= 4(1) - 2(1)^2 \\
 &= 4 - 2 \\
 \Rightarrow \boxed{y = 2}
 \end{aligned}$$

∴ The maximum value = 2

2. Find the minimum value of  $y = x^2 + 4x + 1$ .

**Solution:**

$$\text{Given: } y = x^2 + 4x + 1 \quad \dots\dots\dots (1)$$

$$\frac{dy}{dx} = (2x) + 4(1) + 0 = 2x + 4$$

$$\frac{d^2y}{dx^2} = 2(1) + 0 = 2$$

$$\frac{dy}{dx} = 0 \Rightarrow 2x + 4 = 0 \Rightarrow \boxed{x = -2}$$

$$\text{When } x = -2 \Rightarrow \frac{d^2y}{dx^2} = 2 > 0 (\text{positive})$$

⇒  $x = -2$  is a minimum point.

Put  $x = -2$  in (1) we get,

$$\begin{aligned}
 y &= (-2)^2 + (4)(-2) + 1 \\
 &= 4 - 8 + 1
 \end{aligned}$$

$$y = -3$$

∴ Minimum value = -3

### PART – C

1. Find the maximum and minimum values of  $2x^3 - 15x^2 + 36x + 18$ .

**Solution:**

$$\text{Let } y = 2x^3 - 15x^2 + 36x + 18 \quad \dots\dots\dots (1)$$

$$\frac{dy}{dx} = 2(3x^2) - 15(2x) + 36(1) + 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 6x^2 - 30x + 36} \quad \dots\dots\dots(2)$$

$$\frac{d^2y}{dx^2} = 6(2x) - 30(1) + 0$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} = 12x - 30} \quad \dots\dots\dots(3)$$

For maximum (or) minimum  $\frac{dy}{dx} = 0$

$$\Rightarrow 6x^2 - 30x + 36 = 0$$

$$\div 6, \quad x^2 - 5x + 6 = 0$$

$$(x - 2)(x - 3) = 0$$

$$x - 2 = 0 \quad | \quad x - 3 = 0$$

$$x = 2 \quad | \quad x = 3$$

Case (i): When  $x = 2$ ,  $\frac{d^2y}{dx^2} = 12(2) - 30 = 24 - 30 = -6 < 0$  (negative)

$\Rightarrow x = 2$  is maximum point.

To find maximum value: Put  $x = 2$  in (1)

$$\begin{aligned} y &= 2(2)^3 - 15(2)^2 + 36(2) + 18 \\ &= 2(8) - 15(4) + 36(2) + 18 \\ &= 16 - 60 + 72 + 18 = 46 \end{aligned}$$

Maximum value = 46

Case (ii): When  $x = 3$

$\frac{d^2y}{dx^2} = 12(3) - 30 = 36 - 30 = 6 < 0$  (positive)

$\Rightarrow x = 3$  is minimum point.

To find minimum value: Put  $x = 3$  in (1)

$$\begin{aligned} y &= 2(3)^3 - 15(3)^2 + 36(3) + 18 \\ &= 2(27) - 15(9) + 36(3) + 18 \\ &= 54 - 135 + 108 + 18 = 45 \end{aligned}$$

Minimum value = 45

2. Find the maximum and minimum values of  $(x - 1)^2(x - 2)$ .

**Solution:**

$$\begin{aligned} \text{Let } y &= (x - 1)^2(x - 2) \quad \dots\dots\dots (1) \\ &= (x^2 - 2x + 1)(x - 2) \\ &= x^3 - 2x^2 + x - 2x^2 + 4x - 2 \end{aligned}$$

$$\Rightarrow y = x^3 - 4x^2 + 5x - 2$$

$$\frac{dy}{dx} = (3x^2) - 4(2x) + 5(1) - 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 3x^2 - 8x + 5} \quad \dots\dots\dots(2)$$

$$\frac{d^2y}{dx^2} = 3(2x) - 8(1) + 0$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} = 6x - 8} \quad \dots\dots\dots(3)$$

For maximum (or) minimum  $\frac{dy}{dx} = 0$

$$\begin{aligned} 3x^2 - 8x + 5 &= 0 \\ 3x^2 - 3x - 5x + 5 &= 0 & 3x - 5 = 0 & \left| \begin{array}{l} (x - 1) = 0 \\ x = 1 \end{array} \right. \\ 3x(x - 1) - 5(x - 1) &= 0 & \Rightarrow \boxed{x = \frac{5}{3}} & \\ (3x - 5)(x - 1) &= 0 & & \end{aligned}$$

Case (i): When  $x = \frac{5}{3}$ ,  $\frac{d^2y}{dx^2} = 6\left(\frac{5}{3}\right) - 8 = 10 - 8 = 2 > 0$  (positive)

$\Rightarrow x = \frac{5}{3}$  is a minimum point.

To find minimum value: Put  $x = \frac{5}{3}$  in (1)

$$y = \left(\frac{5}{3} - 1\right)^2 \left(\frac{5}{3} - 2\right) = \left(\frac{2}{3}\right)^2 \left(-\frac{1}{3}\right) = -\frac{4}{27}$$

$$\text{Minimum value} = \frac{-4}{27}$$

Case (ii): When  $x = 1$  then  $\frac{d^2y}{dx^2} = 6(1) - 8 = 6 - 8 = -2 < 0$  (negative)

$\Rightarrow x = 1$  is a maximum point.

To find maximum value: Put  $x = 1$  in (1)

$$y = (1 - 1)^2 (1 - 2) = (0)(-1) = 0$$

Maximum value = 0

3. Show that  $\sin x (1 + \cos x)$  has a maximum when  $x = \frac{\pi}{3}$ .

**Solution:**

$$\text{Let } y = \sin x (1 + \cos x) \quad \dots\dots\dots (1)$$

$$\frac{dy}{dx} = \sin x (0 - \sin x) + (1 + \cos x)(\cos x)$$

$$= -\sin^2 x + \cos x + \cos^2 x$$

$$= (\cos^2 x - \sin^2 x) + \cos x$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \cos 2x + \cos x} \quad \dots\dots\dots (2)$$

$$\frac{d^2y}{dx^2} = -2\sin 2x - \sin x \quad \dots\dots\dots (3)$$

$$\text{For maximum, } \frac{dy}{dx} = 0$$

$$\cos 2x + \cos x = 0$$

$$2\cos \frac{3x}{2}, \cos \frac{x}{2} = 0$$

Here  $2 \neq 0$ ,

When  $x = \frac{\pi}{3}$  then

$$\frac{d^2y}{dx^2} = -2\sin 2\left(\frac{\pi}{3}\right) - \sin \frac{\pi}{3}$$

$$= -2\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2}$$

$$= -\sqrt{3} - \frac{\sqrt{3}}{2} < 0$$

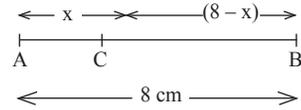
$y$  is maximum when  $x = \frac{\pi}{3}$

$$\begin{array}{l|l} \cos \frac{3x}{2} = 0 & \cos \frac{x}{2} = 0 \\ \cos \frac{3x}{2} = \cos \frac{\pi}{2} & \cos \frac{x}{2} = \cos \frac{\pi}{2} \\ \frac{3x}{2} = \frac{\pi}{2} & \frac{x}{2} = \frac{\pi}{2} \\ x = \frac{\pi}{3} & x = \pi \end{array}$$

4. AB is 8 cm long, find a point C on AB so that  $3AC^2 + BC^2$  may be minimum.

**Solution:**

$$AB = 8 \text{ cm}$$



$$\text{Let } AC = x \text{ cm}$$

$$\Rightarrow BC = (8 - x) \text{ cm}$$

$$\text{Let } y = 3AC^2 + BC^2$$

$$= 3(x)^2 + (8 - x)^2$$

$$= 3x^2 + 64 - 16x + x^2$$

$$\Rightarrow y = 4x^2 - 16x + 64 \quad \dots\dots\dots (1)$$

$$\frac{dy}{dx} = 4(2x) - 16(1) + 0 = 8x - 16 \quad \dots\dots\dots(2)$$

$$\frac{d^2y}{dx^2} = 8(1) - 0 = 8 \quad \dots\dots\dots(3)$$

$$\text{For minimum, } \frac{dy}{dx} = 0$$

$$8x - 16 = 0$$

$$\Rightarrow \boxed{x = 2}$$

$$\text{When } x = 2, \frac{d^2y}{dx^2} = 8 > 0 \text{ (positive).}$$

$\Rightarrow x = 2$  is a minimum point.

$\Rightarrow y = 3AC^2 + BC^2$  is minimum when  $x = 2$ .

When C is 2 cm from A,  $3AC^2 + BC^2$  is minimum.

## EXERCISE

### PART - A

1. If  $s = 2 \sin t$  find velocity and acceleration.
2. The distance travelled by a body is given by  $s = t^3$ , find the velocity and acceleration at 't' sec.
3. If the distance travelled by a particle is  $s = 5t^2 + 7t - 3$  find its initial velocity.
4. If  $s = 4 \cos t + 5 \sin t$ , find its initial velocity.
5. If  $s = 8 \cos 2t + 4 \sin t$ , find its initial velocity.
6. If  $s = 2t^2 - 12t + 3$  find the time when its velocity is zero.
7. If  $s = 7t^2 - 2t + 3$  find its initial acceleration.
8. Find the slope of the tangent to the curve  $y^2 = 4x$  at (1, 2).
9. Find the gradient of the tangent to the curve  $y = 4x - 3x^2$  at (2, -4).
10. Find the slope of the tangent to the curve  $y = x^4 - 2x$  at (1, -1).
11. Find the slope of the normal to the curve  $y^2 = 8x$  at (2, 4).
12. Find the slope of normal to the curve  $y = 3x^2 - 4x$  at (1, -1).
13. Show that the function  $y = \sin x$  is decreasing in  $\left(\frac{\pi}{2}, 2\pi\right)$ .

14. Show that  $y = x^2 - 1$  is an increasing function in the interval  $(0, 2)$ .
15. Show that the function  $y = 2x - x^2$  is maximum at  $x = 1$ .
16. Show that the function  $y = x^2 - 4x$  is minimum at  $x = 2$ .

### PART – B

1. The distance 's' travelled by a particle in 't' sec is given by  $s = 4t^3 - 5t + 6$  find the velocity and acceleration at the end of 10 secs.
2. If  $s = 2t^3 - 3t^2 + 1$  find the velocity and acceleration after  $t = 1$  secs.
3. If the distance time formula is given by  $s = 2t^3 - 15t^2 + 36t + 7$  find the time when the velocity becomes zero.
4. The distance travelled by a particle at time 't' sec is given by  $s = ae^{3t} + be^{-3t}$ . Show that the acceleration is always nine times the distance covered.
5. Find the equation of tangent to the curve  $y = 2 - 3x + 4x^2$  at  $x = 1$ .
6. Find the equation of tangent to the curve  $y = x^2 - x + 1$  at  $(2, 3)$ .
7. Find the equation of normal to the curve  $y = 6 - x + x^2$  at  $(2, 8)$ .
8. Find the equation of normal to the curve  $y = x^2 - 7x + 12$  at  $(3, 0)$ .
9. Find the maximum value of  $y = 2x - 3x^2$ .
10. Find the minimum value of  $y = x^2 - 4x$ .

### PART – C

1. The distance 's' metres at time 't' min of a particle moving in a straight line is given by  $s = 2t^3 - 9t^2 + 12t + 6$ . Find the velocity when acceleration is zero. Find also the acceleration when the velocity is zero.
2. The distance 's' metres travelled by a body in 't' secs is given by the formula.  $S = t^3 - 6t^2 + 12t + 8$ . Find its velocity when acceleration is zero and calculate the acceleration when the vector is zero.
3. A missile is fired from the ground level rises 'x' metres vertically upwards in time 3 secs and  $x = 100t - \frac{25}{2}t^2$ . Find the initial velocity and maximum height of the missile.
4. The distance time formula of a moving point is  $x = a \cos nt + b \sin nt$ . Prove that the acceleration varies as the distance.
5. A body moves in a straight line in such a manner that  $s = \frac{1}{2}vt$ ; 's' being the space travelled in time 't' and 'V' the velocity at the end of 't' secs. Prove that the acceleration is constant.
6. Find the equation of the tangent and normal to the curve  $y = 6 + x - x^2$  at  $(2, 4)$ .
7. Find the equation of the tangent and normal to the curve  $y = 5 - 2x - 3x^2$  at the point  $(2, -11)$ .
8. Find the equation of the tangent and normal at  $(a, a)$  on the curve  $y^2 = \frac{x^3}{2a - x}$ .
9. Find the equation of tangent and normal to the curve  $y = \frac{x+8}{x^2+1}$  at  $(2, 2)$ .
10. Find the equation of tangent and normal to the curve  $y = (x - 2)(x - 5)$  at the point where the curve cuts the x-axis.

11. Find the equation of the tangent and normal to the curve  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  at  $(a \cos \theta, b \sin \theta)$
12. Find the equation of tangent to the curve  $y = x^2 + 5x + 7$  where it cuts the y-axis.
13. Find the maximum and minimum values of
- $y = 4x^3 - 18x^2 + 24x - 7$
  - $y = 2x^3 - 3x^2 - 36x + 10$
  - $y = x^3 - 4x^2 + 5x - 2$
  - $y = 2x^3 + 3x^2 - 36x + 1$
  - $y = (x - 4)(x + 1)^2$
14. Find the maximum value of  $\frac{\log x}{x}$  for positive value of x.
15. Find the minimum value of  $x \log x$ .
16. Find two numbers whose sum is 10 and whose product is maximum.
17. The bending moment at any point at a distance 'x' from one end of a beam of length 'l' uniformly loaded is given by  $M = \frac{1}{2}wx - \frac{1}{2}wx^2$  where 'w' is the load/unit length. Show that the bending moment is maximum at the centre of the beam.
18. If  $s = t^3 - 6t^2 + 42t$ , find the minimum velocity.
19. ABCD is a rectangle in which AB = 9 cm and BC = 6 cm. Find a point P on CD such that  $PA^2 + PB^2$  is minimum.
20. Find two positive numbers whose product is 100 and whose sum is minimum.

### ANSWERS

#### PART - A

- 1)  $v = 2 \cos t, a = -2 \sin t$       2)  $v = 3t^2, a = 6t$       3)  $v = 7$       4)  $v = 5$   
 5)  $v = 4$       6)  $t = 3 \text{ sec}$       7)  $t = 14$       8)  $m = 1$       9)  $m = -8$       10)  $m = 2$   
 11)  $m = -1$       12)  $m = -\frac{1}{2}$ .

#### PART - B

- 1)  $v = 1195 \text{ m/sec}; a = 240 \text{ m/sec}^2$       2)  $v = 0 \text{ unit/sec}; a = 6 \text{ unit/sec}^2$   
 3)  $t = 2, 3$       5)  $5x - y - 2 = 0$   
 6)  $3x - y - 3 = 0$       7)  $x + 3y - 26 = 0$   
 8)  $x - y - 3 = 0$   
 9)  $\left(\frac{1}{3}, \frac{10}{9}\right)$   
 10)  $-4$

**PART – C**

- 1)  $v = -\frac{3}{2}$  unit/min;  $a = \pm 6$  unit/min<sup>2</sup>      2)  $v = 0$ ;  $a = 0$   
3)  $v = 100$  unit/sec;  $h = 200$  m      6)  $3x + y - 10 = 0$ ,  $x - 3y + 10 = 0$   
7)  $14x + y - 17 = 0$ ;  $x - 14y - 156 = 0$       8)  $2x - y - a = 0$ ;  $x + 2y - 3a = 0$   
9)  $7x + 5y - 24 = 0$ ;  $5x - 7y + 4 = 0$   
10)  $x + y - 2 = 0$ ,  $x - y - 2 = 0$  &  $x - y - 3 = 0$ ,  $x + y - 3 = 0$   
11)  $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ ;  $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b$   
12)  $5x - y + 7 = 0$   
13) (i) (3, 1)    (ii) (54, -71)    (iii)  $\left(0, \frac{-4}{27}\right)$     (iv) (82, -43)    (v)  $\left(0, \frac{-500}{27}\right)$   
14)  $\frac{1}{e}$       15)  $-\frac{1}{e}$       16) 5, 5      18) 30 cm/sec      20) 10, 10

# UNIT – IV

## APPLICATION OF INTEGRATION-I

### 4.1 Area and Volume

Area and Volume – Area of circle, Volume of Sphere and cone – Simple Problems.

### 4.2 First Order Differential Equation

Solution of first order variable separable type differential equation – Simple problems.

### 4.2 Linear Type Differential Equation

Solution of linear differential equation – Simple problems.

### Introduction

In Engineering Mathematics-II, we discussed the basic concepts of integration. In Engineering Mathematics-I, we studied the formation of differential equation. In this unit, we shall study the application of integration and first order differential equation.

## 4.1 AREA AND VOLUME

### Area and Volume:

We apply the concept of definite integral to find the area and volume.

### Area:

The area under the curve  $y = f(x)$  between the x-axis and the ordinates  $x = a$  and  $x = b$  is given by the definite integral  $\int_a^b f(x) dx$  (or)  $\int_a^b y dx$ .

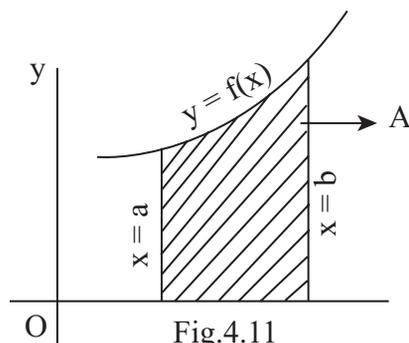


Fig.4.11

The area is shown as shaded region (A) in Fig.4.11

$$\text{Area} = A = \int_a^b f(x) dx = \int_a^b y dx$$

Similarly, the area between the curve  $x = g(y)$ ,  $y$ -axis and the lines  $y = c$  and  $y = d$  shown as shaded region (A), in Fig. 4.12

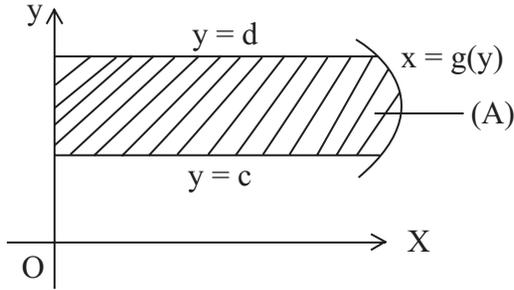


Fig 4.12

is given by

$$\text{Area} = A = \int_c^d g(y) \, dy \text{ (or) } \int_c^d x \, dy$$

**Volume:**

The volume of the solid obtained by rotating the area (shown in Fig.4.11) bounded by the curve  $y = f(x)$ ,  $x$ -axis and the lines  $x = a$  and  $x = b$  is given by

$$V = \pi \int_a^b [f(x)]^2 \, dx \quad \text{(or) } \pi \int_a^b y^2 \, dx$$

Similarly, the volume of solid obtained by rotating the area (shown in Fig.4.12) bounded by the curve  $x = g(y)$ ,  $y$ -axis and the lines  $y = c$  and  $y = d$  about the  $y$ -axis is given by

$$V = \pi \int_c^d g(y)^2 \, dy = \pi \int_c^d x^2 \, dy$$

**4.1 WORKED EXAMPLES**

**PART –A**

1. Find the area bounded by the curve  $y = 4x^3$ , the  $x$ -axis and the ordinates  $x = 0$  and  $x = 1$ .

**Solution:**

$$\begin{aligned} \text{Area} &= \int_a^b y \, dx = \int_0^1 4x^3 \, dx \\ &= \left[ \frac{4x^4}{4} \right]_0^1 = [x^4]_0^1 \\ &= (1)^4 - (0)^4 = 1 - 0 = 1 \text{ square units} \end{aligned}$$

2. Find the area bounded by the curve  $y = e^x$ , the  $x$ -axis and the ordinates  $x = 0$  and  $x = 6$ .

**Solution:**

$$\begin{aligned} \text{Area} &= \int_0^6 e^x \, dx = \int_0^6 e^x \, dx = [e^x]_0^6 \\ &= e^6 - e^0 = e^6 - 1 \text{ square units} \end{aligned}$$

3. Find the area bounded by the curve  $x = 2y^2$ , the y-axis and the lines  $y = 0$  and  $y = 3$ .

**Solution:**

$$\begin{aligned} \text{Area} &= \int_0^3 2y^2 dy = \left[ \frac{2y^3}{3} \right]_0^3 \\ &= \left[ \frac{2(3)^3}{3} \right] - 0 = \frac{54}{3} = 18 \text{ square units} \end{aligned}$$

4. Find the volume of the solid formed when the area bounded by the area  $y^2 = 4x$ , the x-axis and the lines  $x = 0$  and  $x = 1$  is rotated about the x-axis.

**Solution:**

$$\begin{aligned} \text{Volume} = V &= \pi \int_a^b y^2 dx \\ &= \pi \int_0^1 4x dx = \pi \left[ \frac{4x^2}{2} \right]_0^1 \\ &= \pi [2x^2]_0^1 = \pi [2(1)^2] - 0 \\ &= 2\pi \text{ cubic units} \end{aligned}$$

### PART – B

1. Find the area bounded by the curve  $y = x^2 + x + 2$ , x-axis and the lines  $x = 1$  and  $x = 2$ .

**Solution:**

$$\begin{aligned} \text{Area} &= \int_a^b y dx = \int_1^2 (x^2 + x + 2) dx \\ &= \left[ \frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_1^2 \\ &= \left[ \frac{2^3}{3} + \frac{2^2}{2} + 2(2) \right] - \left[ \frac{1^3}{3} + \frac{1^2}{2} + 2(1) \right] \\ &= \left[ \frac{8}{3} + \frac{4}{2} + 4 \right] - \left[ \frac{1}{3} + \frac{1}{2} + 2 \right] \\ &= \left[ \frac{8}{3} + 6 \right] - \left[ \frac{2+3}{6} + 2 \right] \\ &= \left[ \frac{8+18}{3} \right] - \left[ \frac{5}{6} + 2 \right] \\ &= \left[ \frac{26}{3} \right] - \left[ \frac{5+12}{6} \right] = \frac{26}{3} - \frac{17}{6} \\ &= \frac{52-17}{6} = \frac{35}{6} \text{ sq.units} \end{aligned}$$

2. Find the area enclosed by one arch of the curve  $y = \sin x$ ,  $x$ -axis between  $x = 0$  and  $x = \pi$ .

**Solution:**

$$\begin{aligned}\text{Area} &= \int_a^b y \, dx = \int_0^{\pi} \sin x \, dx \\ &= [-\cos x] = [-\cos \pi] - [-\cos 0] \\ &= -(-1) + 1 = 1 + 1 = 2 \text{ square units}\end{aligned}$$

3. Find the volume of the solid generated when the region enclosed by  $y^2 = 4x^3 + 3x^2 + 2x$  between  $x = 1$  and  $x = 2$  is revolved about the  $x$ -axis.

**Solution:**

$$\begin{aligned}\text{Volume} = V &= \pi \int_a^b y^2 \, dx \\ &= \pi \int_1^2 (4x^3 + 3x^2 + 2x) \, dx \\ &= \pi \left[ \frac{4x^4}{4} + \frac{3x^3}{3} + \frac{2x^2}{2} \right]_1^2 \\ &= \pi [x^4 + x^3 + x^2]_1^2 \\ &= \pi [2^4 + 2^3 + 2^2] - \pi [1^4 + 1^3 + 1^2] \\ &= \pi(16 + 8 + 4) - \pi(1 + 1 + 1) \\ &= 28\pi - 3\pi = 25\pi \text{ cubic units}\end{aligned}$$

4. Find the volume of the solid formed when the area bounded by the curve  $y = \sqrt{10+x}$  between  $x = 0$  and  $x = 5$  is rotated about  $x$ -axis.

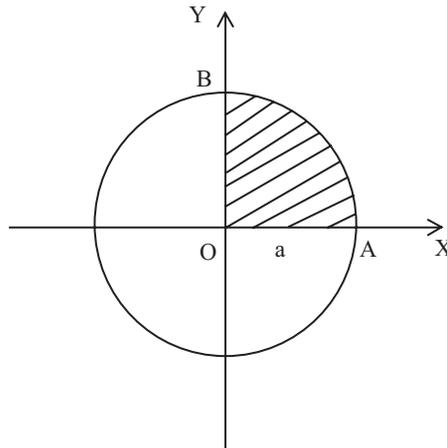
**Solution:**

$$\begin{aligned}\text{Volume} = V &= \pi \int_a^b y^2 \, dx = \pi \int_0^5 [\sqrt{10+x}]^2 \, dx \\ &= \pi \int_0^5 (10+x) \, dx \\ &= \pi \left[ 10x + \frac{x^2}{2} \right]_0^5 \\ &= \pi \left[ 10(5) + \frac{5^2}{2} \right] - \pi \left[ 10(0) + \frac{0^2}{2} \right] \\ &= \pi \left[ 50 + \frac{25}{2} \right] - 0 \\ &= \pi \left[ \frac{100+25}{2} \right] = \frac{125\pi}{2} \text{ cubic units}\end{aligned}$$

**PART – C**

1. Find the area of a circle of radius  $a$ , using integration.

**Solution:**



Area bounded by the circle  $x^2 + y^2 = a^2$ , the x-axis, and the lines  $x = 0$  and  $x = a$  is given by

$$\begin{aligned} \text{Area OAB} &= \int_0^a y \, dx \\ &= \int_0^a \sqrt{a^2 - x^2} \, dx \\ &= \left[ \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a \quad [\text{Formula}] \\ &= \left[ \frac{a}{2} \sqrt{a^2 - a^2} + \frac{a^2}{2} \sin^{-1} \frac{a}{a} \right] - [0 + 0] \end{aligned}$$

$$= \frac{a}{2}(0) + \frac{a^2}{2} \sin^{-1}(1)$$

$$= 0 + \frac{a^2}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi a^2}{4}$$

$\sin 90 = 1$ $90^\circ = \sin^{-1}(1)$ $\frac{\pi}{2} = \frac{180}{2} = \sin^{-1}(1)$
--

Area of the circle =  $4 \times$  Area OAB

$$= 4 \times \frac{\pi a^2}{4}$$

$$= \pi a^2 \text{ sq.units}$$

Aliter:

$$I = \int_0^a \sqrt{a^2 - x^2} \, dx$$

Put  $x = a \sin \theta$

when $x = 0$		when $x = a$
$a \sin \theta = 0$		$a \sin \theta = a$
$\sin \theta = 0 = \sin \theta$		$\sin \theta = 1 = \sin 90 = \sin\left(\frac{\pi}{2}\right)$
$\theta = 0$		$\theta = \frac{\pi}{2}$

$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta}$	
$= \sqrt{a^2(1 - \sin^2 \theta)}$	$x = a \sin \theta$
$= \sqrt{a^2 \cos^2 \theta}$	$\frac{dx}{d\theta} = a \cos \theta$
$= a \cos \theta$	$dx = a \cos \theta d\theta$

$$\begin{aligned} \therefore I &= \int_0^{\pi/2} (a \cos \theta)(a \cos \theta) d\theta \\ &= \int_0^{\pi/2} a^2 \cos^2 \theta d\theta \end{aligned}$$

$$= \int_0^{\pi/2} a^2 \frac{(1 + \cos 2\theta)}{2} d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} (1 + \cos 2\theta) d\theta$$

$$= \frac{a^2}{2} \left[ \theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

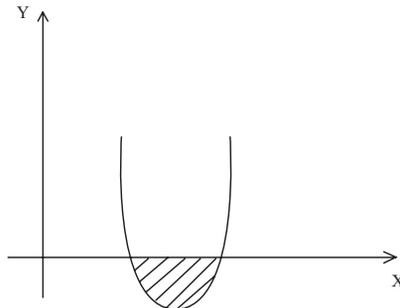
$$= \frac{a^2}{2} \left[ \frac{\pi}{2} + \frac{\sin(2 \times \pi/2)}{2} \right] - \frac{a^2}{2} (0 + 0)$$

$$= \frac{a^2}{2} \left[ \frac{\pi}{2} + \frac{\sin 180}{2} \right] = \frac{a^2}{2} \left[ \frac{\pi}{2} + 0 \right] = \frac{\pi a^2}{4}$$

$$\begin{aligned} &\frac{1 + \cos 2\theta}{2} \\ &= \frac{\cancel{x} + 2 \cos^2 \theta - \cancel{x}}{2} \\ &= \cos^2 \theta \end{aligned}$$

2. Find the area bounded by the curve  $y = x^2 - 6x + 8$  and the x-axis.

**Solution:**



To find limits

Put  $y = 0$  as the curve meets the x-axis

$$x^2 - 6x + 8 = 0$$

$$(x - 2)(x - 4) = 0$$

$$x = 2, 4$$

$$\begin{aligned}
\text{Area} &= \int_2^4 (x^2 - 6x + 8) \, dx \\
&= \left[ \frac{x^3}{3} - \frac{6x^2}{2} + 8x \right]_2^4 \\
&= \left[ \frac{4^3}{3} - \frac{6(4)^2}{2} + 8(4) \right] - \left[ \frac{2^3}{3} - \frac{6(2)^2}{2} + 8(2) \right] \\
&= \left[ \frac{64}{3} - 48 + 32 \right] - \left[ \frac{8}{3} - 12 + 16 \right] \\
&= \left[ \frac{64}{3} - 16 \right] - \left[ \frac{8}{3} + 4 \right] \\
&= \left[ \frac{64 - 48}{3} \right] - \left[ \frac{8 + 12}{3} \right] \\
&= \frac{16}{3} - \frac{20}{3} = \frac{-4}{3} = \frac{4}{3} \text{ sq. units (as Area is positive)}
\end{aligned}$$

3. Find the area bounded by the curve  $y = 10 - 3x - x^2$  and the x-axis.

**Solution:**

To find limits

Put  $y = 0$  as the curve cuts the x-axis

$$10 - 3x - x^2 = 0$$

$$x^2 + 3x - 10 = 0$$

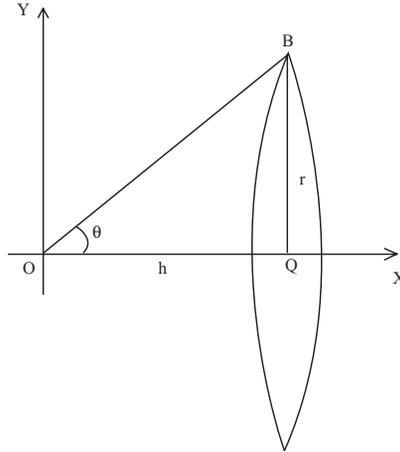
$$(x + 5)(x - 2) = 0$$

$$x = -5, x = 2$$

$$\begin{aligned}
\text{Area} &= \int_{-5}^2 (10 - 3x - x^2) \, dx \\
&= \left[ 10x - \frac{3x^2}{2} - \frac{x^3}{3} \right]_{-5}^2 \\
&= \left[ 10(2) - \frac{3(2)^2}{2} - \frac{2^3}{3} \right] - \left[ 10(-5) - \frac{3(-5)^2}{2} - \frac{(-5)^3}{3} \right] \\
&= \left[ 20 - 6 - \frac{8}{3} \right] - \left[ -50 - \frac{75}{2} + \frac{125}{3} \right] \\
&= \left[ 14 - \frac{8}{3} \right] - \left[ \frac{-100 - 75}{2} + \frac{125}{3} \right] \\
&= \frac{34}{3} - \left[ \frac{-525 + 250}{6} \right] \\
&= \frac{68}{6} - \left[ \frac{-275}{6} \right] \\
&= \frac{68}{6} + \frac{275}{6} = \frac{343}{6} \text{ sq. units}
\end{aligned}$$

4. Find the volume of a right circular cone of height  $h$  and base radius  $r$  by integration.

**Solution:**



Rotate a right angled triangle OAB with sides  $OA = h$ ,  $AB = r$  about the x-axis. Then we get a right circular cone.

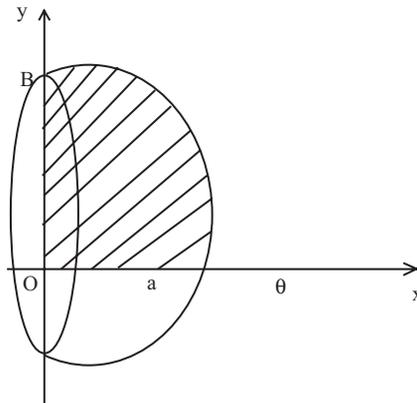
Volume of cone

$$\begin{aligned}
 V &= \pi \int_0^h y^2 dx \\
 &= \pi \int_0^h \frac{r^2}{h^2} x^2 dx \\
 &= \left[ \frac{\pi r^2}{h^2} \frac{x^3}{3} \right]_0^h \\
 &= \left[ \frac{\pi r^2}{h^2} \frac{h^3}{3} \right] - [0] \\
 &= \frac{1}{3} \pi r^2 h
 \end{aligned}$$

<p>Equation of OB is</p> $y = mx$ $= \frac{r}{h} x$ $y^2 = \frac{r^2}{h^2} x^2$
---

5. Find the volume of a sphere of radius 'a' by integration.

**Solution:**



Rotate the area OAB (Quadrant of a circle) about OA, the x-axis. Then we get a hemi-sphere.

## Volume of hemisphere

$$\begin{aligned}
 V &= \pi \int_0^a y^2 dx \\
 &= \pi \int_0^a (a^2 - x^2) dx \\
 &= \pi \left[ a^2 x - \frac{x^3}{3} \right]_0^a \\
 &= \pi \left[ a^2 a - \frac{a^3}{3} \right] = \pi \left[ a^3 - \frac{a^3}{3} \right] \\
 &= \pi \left[ \frac{3a^3 - a^3}{3} \right] = \pi \left[ \frac{2a^3}{3} \right] = \frac{2\pi a^3}{3}
 \end{aligned}$$

$$\text{Volume of sphere} = 2 \times \frac{2\pi a^3}{3} = \frac{4}{3} \pi a^3$$

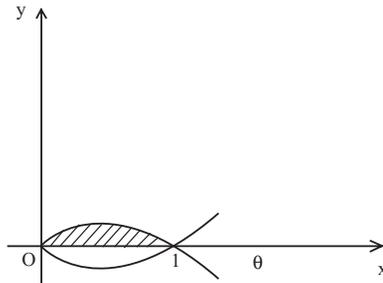
6. Find the volume generated by the area enclosed by the curve  $y^2 = x(x-1)^2$  and the x-axis, when rotated about x-axis.

**Solution:**

To find limits

Put  $y = 0$  as the curve cuts the x-axis.

$$x(x-1)^2 = 0 \Rightarrow x = 0 \text{ (or) } x - 1 = 0 \Rightarrow x = 1.$$



$$\begin{aligned}
 V &= \pi \int_0^1 y^2 dx \\
 &= \pi \int_0^1 x(x-1)^2 dx \\
 &= \pi \int_0^1 x(x^2 - 2x + 1) dx \\
 &= \pi \int_0^1 (x^3 - 2x^2 + x) dx \\
 &= \pi \left[ \frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} \right]_0^1 \\
 &= \pi \left[ \frac{1^4}{4} - \frac{2(1)^3}{3} + \frac{1^2}{2} \right] - [0 - 0 + 0] \\
 &= \pi \left[ \frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right] \\
 &= \pi \left[ \frac{3 - 8 + 6}{12} \right] = \pi \left[ \frac{1}{12} \right] = \frac{\pi}{12} \text{ cubic units}
 \end{aligned}$$

## 4.2 FIRST ORDER DIFFERENTIAL EQUATION

### Introduction:

Since the time of Newton, physical problems have been investigated by formulating them mathematically as differential equations. Many mathematical models in engineering employ differential equations extensively.

In the first order differential equation, say  $\frac{dy}{dx} = f(x, y)$ , it is sometimes possible to group function of  $x$  with  $dx$  on one side and function of  $y$  with  $dy$  on the other side. This type of equation is called variables separable differential equations. The solution can be obtained by integrating both sides after separating the variables.

### 4.2 WORKED EXAMPLES

#### PART – A

1. Solve:  $x dx + y dy = 0$

**Solution:**

$$x dx = -y dy$$

$$\text{Integrating, } \int x dx = -\int y dy + c$$

$$\frac{x^2}{2} = -\frac{y^2}{2} + c$$

2. Solve :  $x dy + y dx = 0$

**Solution:**

$$x dy = -y dx$$

$$\frac{dy}{y} = \frac{-dx}{x}$$

$$\text{Integrating, } \int \frac{dy}{y} = -\int \frac{dx}{x} + c$$

$$\log y = -\log x + c$$

3. Solve:  $x \frac{dy}{dx} = y$

**Solution:**

$$x dy = y dx$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\text{Integrating, } \int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\log y = \log x + c$$

4. Solve:  $\frac{dy}{dx} - y \cos x = 0$

**Solution:**

$$\frac{dy}{dx} = y \cos x$$

$$dy = y \cos x \, dx$$

$$\frac{dy}{y} = \cos x \, dx$$

Integrating,  $\int \frac{dy}{y} = \int \cos x \, dx$

$$\log y = \sin x + c$$

5. Solve:  $\frac{dy}{dx} = e^x$

**Solution:**

$$dy = e^x \, dx$$

Integrating,  $\int dy = \int e^x \, dx$

$$y = e^x + c$$

6. Solve:  $\frac{dy}{dx} = \frac{1}{1+x^2}$

**Solution:**

$$dy = \frac{dx}{1+x^2}$$

Integrating,  $\int dy = \int \frac{dx}{1+x^2}$

$$y = \tan^{-1} x + c$$

## PART - B

1. Solve:  $\frac{dy}{dx} = \frac{x}{1+x^2}$

**Solution:**

$$dy = \frac{x \, dx}{1+x^2}$$

Multiply both sides by 2, we get

$$2dy = \frac{2x \, dx}{1+x^2}$$

Integrating,  $\int 2dy = \int \frac{2x \, dx}{1+x^2}$

Differentiation of  $Dr = 1 + x^2 = 2x = Nr$   
Then result =  $\log Dr = \log (1 + x^2)$

$$2y = \log (1 + x^2) + c$$

2. Solve:  $\frac{dy}{dx} = e^{x-5y}$

**Solution:**

$$dy = e^x e^{-5y} dx$$

$$\frac{dy}{e^{-5y}} = e^x dx$$

$$e^{5y} dy = e^x dx$$

Integrating,  $\int e^{5y} dy = \int e^x dx$

$$\frac{e^{5y}}{5} = e^x + c$$

3. Solve:  $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

**Solution:**

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}}$$

Integrating,  $\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{\sqrt{1-x^2}}$

$$\sin^{-1} y = \sin^{-1} x + c$$

4. Solve:  $\frac{dy}{dx} = \frac{3+x}{3+y}$

**Solution:**

$$(3+y) dy = (3+x) dx$$

Integrating,  $\int (3+y) dy = \int (3+x) dx$

$$3y + \frac{y^2}{2} = 3x + \frac{x^2}{2} + c$$

## PART – C

1. Solve:  $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$ .

**Solution:**

$$\sec^2 x \tan y dx = -\sec^2 y \tan x dy$$

$$\frac{\sec^2 x dx}{\tan x} = -\frac{\sec^2 y dy}{\tan y}$$

Integrating,  $\int \frac{\sec^2 x dx}{\tan x} = -\int \frac{\sec^2 y dy}{\tan y}$

Note :

If  $\frac{d}{dx}(\text{Dr}) = \text{Nr}$ , then the answer is  $\log \text{Dr}$ .

Here,

$$\frac{d}{dx}(\tan x) = \sec^2 x \text{ and } \frac{d}{dy}(\tan y) = \sec^2 y$$

$$\log \tan x = -\log \tan y + \log c$$

$$\log \tan x + \log \tan y = \log c$$

$$\log \tan x \tan y = \log c$$

$$\tan x \tan y = c$$

2. Solve:  $\frac{dy}{dx} = e^{x-y} + 3x^2e^{-y}$

**Solution:**

$$\frac{dy}{dx} = e^x e^{-y} + 3x^2 e^{-y}$$

$$\frac{dy}{dx} = e^{-y} (e^x + 3x^2)$$

$$dy = e^{-y} (e^x + 3x^2) dx$$

$$\frac{dy}{e^{-y}} = (e^x + 3x^2) dx$$

$$e^y dy = (e^x + 3x^2) dx$$

$$\text{Integrating, } \int e^y dy = \int (e^x + 3x^2) dx$$

$$e^y = e^x + \frac{3x^3}{3} + c$$

$$e^y = e^x + x^3 + c$$

3. Solve :  $(1 - e^x) \sec^2 y dy + e^x \tan y dx = 0$

**Solution:**

$$(1 - e^x) \sec^2 y dy = -e^x \tan y dx$$

$$\frac{\sec^2 y dy}{\tan y} = \frac{-e^x dx}{1 - e^x}$$

$$\text{Integrating, } \int \frac{\sec^2 y dy}{\tan y} = \int \frac{-e^x dx}{1 - e^x}$$

$$\text{As } \frac{d}{dy}(\tan y) = \sec^2 y \text{ and } \frac{d}{dx}(1 - e^x) = -e^x$$

$$\log \tan y = \log (1 - e^x) + \log c$$

$$\log (\tan y) - \log (1 - e^x) = \log c$$

$$\log \frac{\tan y}{1 - e^x} = \log c$$

$$\frac{\tan y}{1 - e^x} = c$$

$$\tan y = c(1 - e^x)$$

4. Solve:  $(x^2 - y) dx + (y^2 - x) dy = 0$

**Solution:**

$$x^2 dx - y dx + y^2 dy - x dy = 0$$

$$x^2 dx + y^2 dy = x dy + y dx$$

Note :

By uv rule,  $d(xy) = x dy + y dx$

$$x^2 dx + y^2 dy = d(xy)$$

$$\text{Integrating, } \int x^2 dx + \int y^2 dy = \int d(xy)$$

$$\frac{x^3}{3} + \frac{y^3}{3} = xy + c$$

### 4.3 LINEAR DIFFERENTIAL EQUATION

A first order differential equation is said to be linear in  $y$  if the power of terms  $\frac{dy}{dx}$  and  $y$  are unity. Differential equations of the form  $\frac{dy}{dx} + Py = Q$ , where  $P$  and  $Q$  are functions of  $x$  are called linear Differential Equations (LDE). The solution of linear differential equation is given by

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

(or) shortly  $y$  (I.F) =  $\int Q$  (IF)  $dx + c$  where IF =  $e^{\int P dx}$ . IF is called Integrating Factor.

Note :

$$e^{\log(f(x))} = f(x)$$

Examples :

$$e^{\log x} = x ; e^{\log x^3} = x^3 ; e^{\log(\sin x)} = \sin x$$

$$e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x} ; e^{-\log(\sin x)} = e^{\log(\sin x)^{-1}} = \text{cosec } x$$

### 4.3 WORKED EXAMPLES

#### PART – A

1. Find the integrating factor of  $\frac{dy}{dx} + \frac{5}{x}y = x$ .

**Solution:**

$$\text{Compare with } \frac{dy}{dx} + Py = Q$$

$$\text{Here } P = \frac{5}{x}$$

$$\begin{aligned} \text{IF} &= e^{\int P dx} = e^{\int \frac{5}{x} dx} = e^{5 \log x} \\ &= e^{\log x^5} = x^5 \end{aligned}$$

2. Find the integrating factor of  $\frac{dy}{dx} + \frac{2x}{1+x^2}y = x^3$ .

**Solution:**

Compare with  $\frac{dy}{dx} + Py = Q$

$$P = \frac{2x}{1+x^2}$$

$$\begin{aligned} \text{IF} &= e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx} \\ &= e^{\log(1+x^2)} = 1+x^2 \end{aligned}$$

3. Find the integrating factor of  $\frac{dy}{dx} - \frac{3}{x}y = x^3 \cos x$ .

**Solution:**

Compare with

$$\frac{dy}{dx} + Py = Q$$

$$P = -\frac{3}{x}$$

$$\therefore \text{IF} = e^{\int P dx} = e^{\int \frac{-3}{x} dx} = e^{-3 \log x} = e^{\log x^{-3}} = x^{-3} = \frac{1}{x^3}$$

4. Find the integrating factor of  $\frac{dy}{dx} = -\frac{y}{x}$ .

**Solution:**

The given equation is

$$\frac{dy}{dx} + \frac{y}{x} = 0$$

$$\text{Here } P = \frac{1}{x}$$

$$\therefore \text{IF} = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

5. Find the integrating factor of  $\frac{dy}{dx} + \frac{1}{1+x^2}y = 1$ .

**Solution:**

$$\text{Here } \frac{1}{1+x^2}$$

$$\therefore \text{IF} = e^{\int P dx} = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1} x}$$

## PART – B

1. Find the integrating factor of  $\frac{dy}{dx} + y \tan x = \sec^2 x$ .

**Solution:**

Compare with  $\frac{dy}{dx} + Py = Q$

Here  $P = \tan x$

$$IF = e^{\int P dx} = e^{\int \tan x dx}$$

Note :

$$\frac{d}{dx} \log(\sec x) = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$$

$$\therefore \int \tan x dx = \log \sec x$$

$$IF = e^{\log \sec x} = \sec x$$

2. Find the integrating factor of  $\frac{dy}{dx} - y \tan x = \cot x$ .

**Solution:**

Here  $P = -\tan x$

$$IF = e^{\int P dx} = e^{\int -\tan x dx}$$

$$= e^{-\log(\sec x)} = e^{\log(\sec x)^{-1}}$$

$$= (\sec x)^{-1} = \frac{1}{\sec x} = \cos x$$

3. Find the integrating factor of  $\frac{dy}{dx} + y \cot x = \sin x$ .

**Solution:**

Here  $P = \cot x$

$$IF = e^{\int P dx} = e^{\int \cot x dx}$$

Note :

$$\frac{d}{dx} \log(\sin x) = \frac{1}{\sin x} \cos x = \cot x$$

$$\therefore \int \cot x dx = \log(\sin x)$$

$$IF = e^{\log(\sin x)} = \sin x$$

4. Find the integration factor of  $\frac{dy}{dx} - y \cot x = 4x^3$ .

**Solution:**

Here  $P = -\cot x$

$$IF = e^{\int P dx} = e^{\int -\cot x dx}$$

$$= e^{-\log(\sin x)} = e^{\log(\sin x)^{-1}}$$

$$= (\sin x)^{-1} = \frac{1}{\sin x} = \operatorname{cosec} x$$

## PART – C

1. Solve:  $\frac{dy}{dx} - \frac{3}{x}y = x^3 \cos x$

**Solution:**

Compare with  $\frac{dy}{dx} + Py = Q$

Here  $P = -\frac{3}{x}$  and  $Q = x^3 \cos x$

$$\begin{aligned} \text{IF} &= e^{\int P dx} = e^{\int -\frac{3}{x} dx} = e^{-3 \log x} \\ &= e^{\log x^{-3}} = x^{-3} = \frac{1}{x^3} \end{aligned}$$

The required solution is

$$\begin{aligned} y(\text{IF}) &= \int Q(\text{IF}) dx + c \\ y\left(\frac{1}{x^3}\right) &= \int x^3 \cos x \frac{1}{x^3} dx + c \\ &= \int \cos x dx + c \\ \frac{y}{x^3} &= \sin x + c \end{aligned}$$

2. Solve:  $\frac{dy}{dx} + \frac{3x^2 y}{1+x^3} = \frac{2}{1+x^3}$

**Solution:**

Here  $P = \frac{3x^2}{1+x^3}$  and  $Q = \frac{2}{1+x^3}$

$$\text{IF} = e^{\int P dx} = e^{\int \frac{3x^2 dx}{1+x^3}}$$

Note:  $\frac{d}{dx}(1+x^3) = 3x^2 \therefore \text{Ans} = \log(1+x^3)$

$$\text{IF} = e^{\log(1+x^3)} = 1+x^3$$

The required solution is

$$\begin{aligned} y(\text{IF}) &= \int Q(\text{IF}) dx + c \\ y(1+x^3) &= \int \frac{2}{(1+x^3)}(1+x^3) dx + c \\ y(1+x^3) &= \int 2 dx + c \\ y(1+x^3) &= 2x + c \end{aligned}$$

3. Solve:  $(1+x^2)\frac{dy}{dx} + 2xy = 1$

**Solution:**

Divide both sides by  $(1+x^2)$

$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{1+x^2}$$

Here  $P = \frac{2x}{1+x^2}$  and  $Q = \frac{1}{1+x^2}$

$$\begin{aligned} \text{IF} &= e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx} \\ &= e^{\log(1+x^2)} \quad \left[ \because \frac{d}{dx}(1+x^2) = 2x \right] \\ &= 1+x^2 \end{aligned}$$

The required solution is

$$y(\text{IF}) = \int Q(\text{IF}) dx + c$$

$$\begin{aligned} y(1+x^2) &= \int \frac{1}{(1+x^2)}(1+x^2) dx + c \\ &= \int dx + c \end{aligned}$$

$$y(1+x^2) = x + c$$

4. Solve:  $\frac{dy}{dx} + y \cot x = 2 \cos x$

**Solution:**

Here  $P = \cot x$  and  $Q = 2 \cos x$

$$\begin{aligned} \text{IF} &= e^{\int P dx} = e^{\int \cot x dx} \quad \left[ \because \frac{d}{dx} \log(\sin x) = \frac{1}{\sin x} \cos x = \cot x \right] \\ &= e^{\log(\sin x)} \\ &= \sin x \end{aligned}$$

The required solution is

$$y(\text{IF}) = \int Q(\text{IF}) dx + c$$

$$\begin{aligned} y(\sin x) &= \int 2 \cos x \sin x dx + c \\ &= \int \sin 2x dx \quad \left[ \because \sin 2A = 2 \sin A \cos A \right] \end{aligned}$$

$$y \sin x = \frac{-\cos 2x}{2} + c$$

5. Solve:  $\frac{dy}{dx} - y \tan x = e^x \sec x$

**Solution:**

Here  $P = -\tan x$  and  $Q = e^x \sec x$

$$IF = e^{\int P dx} = e^{\int -\tan x dx}$$

$$= e^{-\log(\sec x)}$$

$$= e^{\log(\sec x)^{-1}}$$

$$= (\sec x)^{-1} = \frac{1}{\sec x} = \cos x$$

$$\because \frac{d}{dx} \log(\sec x) = \frac{1}{\sec x} \sec x \tan x = \tan x$$

The required solution is

$$y(IF) = \int Q(IF) dx + c$$

$$y(\cos x) = \int e^x \sec x \cos x dx + c$$

$$= \int e^x \frac{1}{\cos x} \cos x dx + c$$

$$y(\cos x) = \int e^x dx + c$$

$$y \cos x = e^x + c$$

## EXERCISE

### PART - A

- Find the area bounded by the curve  $y = 2x$ , the  $x$ -axis and the lines  $x = 0$  and  $x = 1$ .
- Find the area bounded by the curve  $y = x^2$ ,  $x$ -axis between  $x = 0$  and  $x = 2$ .
- Find the area bounded by the curve  $y = \frac{x^2}{2}$ ,  $x$ -axis between  $x = 1$  and  $x = 3$ .
- Find the area bounded by the curve  $xy = 1$ , the  $y$ -axis and the lines  $y = 1$  and  $y = 5$ .
- Find the area bounded by the curve  $x = 2y$ , the  $y$ -axis and the lines  $y = 1$  and  $y = 2$ .
- Solve:  $xy \frac{dy}{dx} = 1$
- Solve:  $e^x dx + e^y dy = 0$
- Solve:  $\frac{dy}{dx} = e^{3x}$
- Solve:  $\frac{dy}{dx} = \frac{\cos x}{y^2}$
- Find the integrating factor of  $\frac{dy}{dx} + 3y = 6$ .
- Find the integrating factor of  $\frac{dy}{dx} + y \sin x = 0$ .
- Find the integrating factor of  $\frac{dy}{dx} + \frac{y}{x} = x$ .

**PART – B**

- Find the area bounded by the curve  $y = x^2 + x$ , x-axis and the lines  $x = 0$  and  $x = 4$ .
- Find the area bounded by the curve  $y = 3x^2 - x$ , x-axis and the ordinates  $x = 0$  and  $x = 6$ .
- Find the volume of the solid generated when the area bounded by the curve  $y^2 = 25x^3$  between  $x=1$  and  $x = 3$  is rotated about x-axis.
- Find the volume of the solid formed when the area bounded by the curve  $y^2 = 8x$  between  $x = 0$  and  $x = 2$  is rotated about x-axis.
- Find the volume of the solid formed when the area bounded by the curve  $x^2 = 3y^2$  between  $y = 0$  and  $y = 1$  is rotated about the y-axis.
- Solve:  $\tan x \sec^2 y \, dy + \tan y \sec^2 x \, dx = 0$
- Solve:  $\frac{dy}{dx} + \frac{1+x^2}{1+y^2} = 0$
- Solve:  $(1+x^2) \sec^2 y \, dy = 2x \tan y \, dx$
- Solve:  $\frac{dy}{dx} = e^{2x+3y}$
- Solve:  $\sec x \, dy + \sec y \, dx = 0$

Find the integrating for the following linear differential equations:

- $\frac{dy}{dx} + y \cot x = x \operatorname{cosec} x.$
- $\frac{dy}{dx} - y \cot x = \sin x.$
- $\frac{dy}{dx} + y \tan x = e^x \cos x.$
- $\frac{dy}{dx} - y \tan x = x^3.$
- $\frac{dy}{dx} + \frac{xy}{1+x^2} = \frac{1}{1+x^2}$

**PART – C**

- Find the area enclosed by  $y = 6 + x - x^2$  and the x-axis.
- Find the area bounded by the curve  $y = x + \sin x$ , the x-axis and the ordinates  $x = 0$  and  $x = \frac{\pi}{2}$ .
- Find the area bounded by the curve  $y = x^2 + x + 1$ , x-axis and the ordinates  $x = 1$  and  $x = 3$ .
- Find the volume of the solid formed when the area of the loop of the curve  $y^2 = 4x(x-1)^2$  rotates about the x-axis.
- Find the volume of the solid bound when the area bounded by the curve  $y^2 = 2 + x - x^2$ , the x-axis and the lines  $x = -1$  and  $x = 2$  is rotated about x axis.
- Find the volume of the solid obtained by revolving  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  about the x-axis.
- Solve:  $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$
- Solve:  $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$
- Solve:  $\frac{dy}{dx} = \frac{1 + \cos y}{1 + \cos x}$

10. Solve:  $\frac{dy}{dx} = \frac{4+y^2}{\sqrt{4-x^2}}$

Solve the following Linear Differential Equations:

11.  $\frac{dy}{dx} + y \cot x = e^x \operatorname{cosec} x.$

12.  $\frac{dy}{dx} - \frac{2y}{x} = x^2 \sin x$

13.  $(1+x^2) \frac{dy}{dx} - 2xy = (1+x^2)^2$

14.  $\frac{dy}{dx} - \frac{3y}{x} = x^3 e^{2x}.$

15.  $2 \cos x \frac{dy}{dx} + 4y \sin x = \sin 2x.$

16.  $\frac{dy}{dx} + y \tan x = \cos^3 x.$

### ANSWERS

#### PART - A

1) 1    2)  $\frac{8}{3}$     3)  $\frac{13}{3}$     4)  $\log 5$     5) 3    6)  $\frac{y^2}{2} = \log x + c$     7)  $e^x + e^y = c$

8)  $y = \frac{e^{3x}}{3}$     9)  $\frac{y^3}{3} = \sin x + c$     10)  $e^{3x}$     11)  $e^{-\cos x}$     12)  $x$

#### PART - B

1)  $\frac{88}{3}$     2) 198    3)  $500\pi$     4)  $16\pi$     5)  $\pi$

6)  $\log \tan y = -\log \tan x + c$  (or)  $\tan x \tan y = c$     7)  $y + \frac{y^3}{3} + x + \frac{x^3}{3} = c$

8)  $\log \tan y = \log(1+x^2) + c$     9)  $\tan y = c(1+x^2)$     10)  $\sin y + \sin x = c$

11)  $\sin x$     12)  $\operatorname{cosec} x$     13)  $\sec x$     14)  $\cos x$     15)  $\sqrt{1+x^2}$

#### PART - C

1)  $\frac{125}{6}$     2)  $\frac{\pi^2}{8}$     3)  $\frac{44}{3}$     4)  $2\pi$     5)  $\frac{9}{2}$     6)  $\frac{4\pi}{3} ab^2$

7)  $e^y = e^x + \frac{x^3}{y}$     8)  $\log \tan y - \log(1-e^x) = c$  (or)  $\tan y = c(1-e^x)$

9)  $2 \tan\left(\frac{y}{2}\right) = -2 \operatorname{cosec}\left(\frac{x}{2}\right) + c$  (or)  $\tan \frac{y}{2} + \operatorname{cosec} \frac{x}{2} = c$     10)  $\frac{1}{2} \tan^{-1}\left(\frac{y}{2}\right) = \sin^{-1}\left(\frac{x}{2}\right) + c$

11)  $y \sin x = e^x + c$     12)  $\frac{y}{x^2} = -\cos x + c$     13)  $\frac{y}{1+x^2} = x + \frac{x^3}{3} + c$     14)  $\frac{y}{x^3} = \frac{e^{2x}}{2} + c$

15)  $y \sec^2 x = \sec x + c$     16)  $y \sec x = \frac{1}{2} \left[ x + \frac{\sin 2x}{2} \right] + c$

# UNIT – V

## APPLICATION OF INTEGRATION-II

### 5.1 SECOND ORDER DIFFERENTIAL EQUATION – I

Solution of second order differential equations of the form  $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$ , where a, b, c, are constants

### 5.2 SECOND ORDER DIFFERENTIAL EQUATION – II

Solution of second order differential equations of the form  $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$ , where a, b, c, are constants and  $f(x) = e^{mx}$ . Simple Problems.

### 5.3 SECOND ORDER DIFFERENTIAL EQUATION – III

Solution of second order differential equations of the form  $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$ , where a, b, c, are constants and  $f(x) = \sin mx$  or  $\cos mx$ . Simple Problems.

## 5.1 Solution of second order differential equations of the

**form  $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$ , where a, b, c, are constants**

Consider the differential equation

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0 \quad \dots\dots\dots(1)$$

Applying the Notation  $\frac{d}{dx} = D$  and  $\frac{d^2}{dx^2} = D^2$  in (1),

We get

$$aD^2y + bDy + cy = 0, D, D^2 \text{ are called differential equation.}$$

$$\text{i.e. } (aD^2 + bD + c) y = 0 \quad \dots\dots\dots (2)$$

Let  $y = e^{mx}$  be a trial solution of (2)

$$\text{Then, } Dy = \frac{d}{dx}(e^{mx}) = m e^{mx} \text{ \& } D^2y = \frac{d^2}{dx^2}(e^{mx}) = m^2 e^{mx} .$$

$$\therefore (2) \text{ becomes } e^{mx} (am^2 + bm + c) = 0$$

Hence m satisfies the quadratic equation

$$am^2 + bm + c = 0$$

This quadratic equation in m is called auxillary equation.

The Auxillary Equation (A.E) of the differential equation (2) is  $am^2 + bm + c = 0 \dots\dots\dots (3)$

Case (i): If the roots of the Auxillary Equation (3) are real and distinct (unequal), say  $m_1$  &  $m_2$ ,  $m_1 \neq m_2$ , then the solution of (2) is

$$y = Ae^{m_1x} + Be^{m_2x}, \text{ where } A \text{ \& } B \text{ are constants.}$$

Case (ii): If the roots of the Auxillary Equation (3) are real and equal, say  $m = m_1 = m_2$ , the solution of (2) is

$$y = (Ax + B)e^{mx}, \text{ where } A \text{ and } B \text{ are constants.}$$

Case (iii): If the roots of the Auxillary Equation (3) are imaginary, say  $m_1 = \alpha + i\beta$  and  $m_2 = \alpha - i\beta$ , then the solution of (2) is

$$y = e^{\alpha x} (A \cos \beta x + B \sin \beta x), \text{ where } A \text{ \& } B \text{ are constants.}$$

## WORKED EXAMPLES

### PART – A

1. If 3 and 4 are the roots of the auxillary equation of second order differential equation with R.H.S zero, write solution of Differential equation.

**Solution:**

The solution is  $y = Ae^{m_1x} + Be^{m_2x}$

$$\text{i.e, } y = Ae^{3x} + Be^{4x}$$

2. If  $-2, -2$  are the roots of the auxillary equation of second order differential equation with R.H.S zero, write solution of Differential equation.

**Solution:**

The solution is  $y = (Ax + b) e^{mx}$

$$\text{i.e } y = (Ax + B) e^{-2x}$$

3. If  $2 \pm 3i$  are the roots of the auxillary equation of second order differential equation with R.H.S zero, write solution of Differential equation.

**Solution:**

The roots of AE are  $2 \pm 3i$  (imaginary roots). Here, real part  $\alpha = 2$  and imaginary part  $\beta = 3$ .

The solution is  $y = e^{\alpha x} (A \cos \beta x + B \sin \beta x)$

$$\text{i.e } y = e^{2x} (A \cos 3x + B \sin 3x)$$

4. Solve the differential equation  $\frac{d^2y}{dx^2} - 4y = 0$ .

**Solution:**

DE is  $(D^2 - 4) y = 0$

Auxillary equation is  $m^2 - 4 = 0$

$$\text{i.e., } m^2 - 4 = 0$$

$$\therefore m = \pm 2$$

$\therefore$  The roots are  $m = 2$  or  $m = -2$  (real & distinct roots)

Solution is  $y = Ae^{m_1x} + Be^{m_2x}$

$$\text{i.e., } y = Ae^{2x} + Be^{-2x}$$

5. Solve:  $y'' - 2y' + y = 0$

**Solution:**

The D.E is  $(D^2 - 2D + 1) y = 0$

A.E is  $m^2 - 2m + 1 = 0$

i.e.,  $(m - 1)^2 = 0$

$m = 1, 1$  (real and equal roots)

Solution is  $y = (Ax + B) e^{mx}$

i.e.,  $y = (Ax + B) e^{1x}$   
 $= (Ax + B) e^x$

6. Solve:  $y_2 + y = 0$

**Solution:**

DE is  $(D^2 + 1) y = 0$

AE is  $m^2 + 1 = 0$

i.e.,  $m^2 = -1$

$m = \pm \sqrt{-1} = \pm i$   
 $= 0 \pm i$  [imaginary roots,  $\alpha = 0, \beta = 1$ ]

$\therefore$  Solution is  $y = e^{\alpha x} (A \cos \beta x + B \sin \beta x)$   
 $= e^{0x} (A \cos 1x + B \sin 1x)$   
 $= 1 (A \cos x + B \sin x)$   
 $\therefore y = A \cos x + B \sin x$

## PART – B

1. Solve:  $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

**Solution:**

DE is  $(D^2 + 5D + 6) y = 0$

The Auxillary equation is

$m^2 + 5m + 6 = 0$

i.e.,  $m^2 + 2m + 3m + 6 = 0$

i.e.,  $m(m + 2) + 3(m + 2) = 0$

i.e.,  $(m + 2)(m + 3) = 0$

$\therefore m = -2, -3$  (real and distinct roots)

$\therefore$  Solution is  $y = Ae^{m_1x} + Be^{m_2x}$   
i.e.,  $y = Ae^{-2x} + Be^{-3x}$

2. Solve:  $y_2 - 6y_1 + 9y = 0$

**Solution:**

DE is  $(D^2 - 6D + 9) y = 0$

AE is  $m^2 - 6m + 9 = 0$

i.e.,  $m^2 - 3m - 3m + 9 = 0$

i.e.,  $m(m - 3) - 3(m - 3) = 0$

i.e.,  $(m - 3)(m - 3) = 0$

$\therefore m = 3, 3$  (real and equal roots)

Solution is  $y = (Ax + B) e^{mx}$   
i.e.,  $y = (Ax + B) e^{3x}$

3. Solve:  $4y'' + 12y' + 9y = 0$

**Solution:**

$$\text{DE is } (4D^2 + 12D + 9) = 0$$

$$\text{AE is } 4m^2 + 12m + 9 = 0$$

$$\text{i.e., } 4m^2 + 6m + 6m + 9 = 0$$

$$\text{i.e., } 2m(2m + 3) + 3(2m + 3) = 0$$

$$\text{i.e., } (2m + 3)(2m + 3) = 0$$

$$\therefore m = -\frac{3}{2}, -\frac{3}{2} \text{ (real and equal parts)}$$

$$\therefore \text{Solution is } y = (Ax + B) e^{mx}$$

$$\text{i.e., } y = (Ax + B) e^{-\frac{3}{2}x}$$

4. Solve:  $(D^2 + D + 1) y = 0$

**Solution:**

$$\text{DE is } (D^2 + D + 1) y = 0$$

$$\text{AE is } m^2 + m + 1 = 0$$

$$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad [a = 1, b = 1, c = 1]$$

$$= \frac{(-1) \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{i^2 \cdot 3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$= \left(-\frac{1}{2}\right) \pm i\left(\frac{\sqrt{3}}{2}\right)$$

$$\left[ \text{imaginary roots, } \alpha = -\frac{1}{2}, \beta = \frac{\sqrt{3}}{2} \right]$$

$$\therefore \text{Solution is } y = e^{\alpha x} (A \cos \beta x + B \sin \beta x)$$

$$\text{i.e., } y = e^{-\frac{1}{2}x} \left( A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x \right)$$

### PART – C

1. Solve:  $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$ , if  $y = 3$  and  $\frac{dy}{dx} = 0$ , when  $x = 0$ .

**Solution:**

$$\text{DE is } (D^2 + D - 2) y = 0$$

$$\text{AE is } m^2 + m - 2 = 0$$

$$\text{i.e., } m^2 + 2m - m - 2 = 0$$

$$\text{i.e., } m(m + 2) - 1(m + 2) = 0$$

$$\text{i.e., } (m + 2)(m - 1) = 0$$

$$\therefore m = -2, 1 \text{ (real and distinct roots)}$$

$$\text{Solution is } y = Ae^{m_1x} + Be^{m_2x}$$

$$\text{i.e., } y = Ae^{-2x} + Be^x \quad \dots\dots\dots(1)$$

Differentiating (1) w.r.t x,

$$\frac{dy}{dx} = -2Ae^{-2x} + Be^x \quad \dots\dots\dots (2)$$

By given,  $y = 3$  when  $x = 0$

$$\begin{aligned} \text{Put } y = 3, x = 0 \text{ in (1)} \quad 3 &= Ae^{-2 \cdot 0} + Be^0 \\ &= A \cdot 1 + B \cdot 1 \end{aligned}$$

$$\text{i.e.,} \quad A + B = 3 \quad \dots\dots\dots (3)$$

By given,  $\frac{dy}{dx} = 0$ , when  $x = 0$

$$\text{Put } x = 0, \frac{dy}{dx} = 0 \text{ in (2)}$$

$$\begin{aligned} 0 &= -2A \cdot e^{-2 \cdot 0} + Be^0 \\ &= -2A \cdot 1 + B \cdot 1 \end{aligned}$$

$$\text{i.e.,} \quad -2A + B = 0 \quad \dots\dots\dots (4)$$

$$(3) - (4) \text{ gives, } 3A = 3 \quad \therefore A = 1$$

$$\text{From (3), } B = 3 - A = 3 - 1 = 2$$

$\therefore$  From (1), solution is

$$y = 1 \cdot e^{-2x} + 2 \cdot e^x$$

$$\text{i.e., } y = e^{-2x} + 2e^x$$

2. Solve  $(D^2 + 36)y = 0$  when  $x = 0, y = 2$  and when  $x = \frac{\pi}{2}, \frac{dy}{dx} = 3$ .

**Solution:**

DE is  $(D^2 + 36)y = 0$

A.E is  $m^2 + 36 = 0$

$$\text{i.e., } m^2 = -36$$

$$\therefore m = \pm\sqrt{-36} = \pm 6i$$

i.e.,  $m = 0 \pm 6i$  (imaginary roots  $\alpha = 0, \beta = 6$ )

$\therefore$  Solution is  $y = e^{\alpha x} (A \cos \beta x + B \sin \beta x)$

$$\text{i.e., } y = e^{0x} (A \cos 6x + B \sin 6x)$$

$$\text{i.e., } y = A \cos 6x + B \sin 6x \quad \dots\dots\dots (1)$$

Differentiation (1) w.r.t x

$$\frac{dy}{dx} = -6A \sin 6x + 6B \cos 6x \quad \dots\dots\dots (2)$$

By given, when  $x = 0, y = 2$

$$\begin{aligned} \therefore \text{From (1), } 2 &= A \cos 6 \cdot 0 + B \sin 6 \cdot 0 \\ &= A(1) + B(0) \end{aligned}$$

$$\therefore A = 2$$

By given, when  $x = \frac{\pi}{2}, \frac{dy}{dx} = 3$

$$\text{From (2), } 3 = -6A \sin 6 \cdot \frac{\pi}{2} + 6B \cos 6 \cdot \frac{\pi}{2}$$

$$= -6A \sin 3\pi + 6B \cos 3\pi$$

$$= -6A(0) + 6B(-1)$$

$$\text{i.e., } -6B = 3 \quad \therefore B = -\frac{3}{6} = -\frac{1}{2}$$

$$\therefore \text{Solution is } y = 2 \cos 6x - \frac{1}{2} \sin 6x \quad \text{[From (1)]}$$

## 5.2 Solution of second order differential equations of the form $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = e^{mx}$ , where $a, b, c$ , are constants

Consider the DE  $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$  .....(1)

There are two solutions for this DE namely

(i) Complementary Function & (ii) Particular Integral

∴ The whole solution for (1) is

$y = \text{Complementary Function} + \text{Particular Integral}$

i.e,  $y = \text{C.F} + \text{P.I}$

The solution of the DE  $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$  is called Complementary Function.

[Ref.5.1 Problems]

To find Particular Integral for the D.E

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = e^{mx}$$

i.e,  $(aD^2 + bD + C) y = e^{mx}$

$$\begin{aligned} \text{P.I} &= \frac{1}{aD^2 + bD + C} (e^{mx}) \\ &= \frac{1}{f(D)} (e^{mx}) \end{aligned}$$

Case (i)

If  $f(m) \neq 0$ ,

$$\begin{aligned} \text{P.I} &= \frac{1}{f(D)} (e^{mx}) \\ &= \frac{e^{mx}}{f(m)} \end{aligned}$$

Case (ii)

If  $f(m) = 0$  and  $f'(m) \neq 0$ ,

$$\begin{aligned} \text{P.I} &= \frac{1}{f(D)} (e^{mx}) \\ &= \frac{x e^{mx}}{f'(m)} \end{aligned}$$

Case (iii)

If  $f(m) = 0$ ,  $f'(m) = 0$  and  $f''(m) \neq 0$

$$\text{P.I} = \frac{x^2 e^{mx}}{f''(m)}$$

## WORKED EXAMPLES PART – A

1. Find the C.F of the DE  $(D^2 + 4)y = e^{2x}$ .

**Solution:**

$$\text{DE is } (D^2 + 4)y = e^{2x}$$

$$\text{AE is } m^2 + 4 = 0$$

$$\text{i.e., } m^2 = -4$$

$$\therefore m = \pm\sqrt{-4} = \pm\sqrt{2i^2} \\ = 0 \pm i(2) \quad (\text{imaginary roots } \alpha = 0, \beta = 1)$$

$$\text{C.F} = e^{\alpha x} (A \cos \beta x + B \sin \beta x)$$

$$= e^{0x} (A \cos 2x + B \sin 2x)$$

$$\text{i.e., C.F} = A \cos 2x + B \sin 2x$$

2. Find the C.F of  $(D^2 - 60D + 800) y = e^{40x}$

**Solution:**

$$\text{D.E is } (D^2 - 60D + 800) y = e^{40x}$$

$$\text{A.E is } m^2 - 60m + 800 = 0$$

$$\text{i.e., } (m - 40)(m - 20) = 0$$

$$\text{i.e., } m = 40, m = 20 \quad (\text{real \& distinct roots})$$

$$\therefore \text{C.F} = y = Ae^{m_1x} + Be^{m_2x} = Ae^{40x} + Be^{20x}$$

## PART – B

1. Find the P.I of D.E  $(D^2 + 7D + 14) y = 8e^{-x}$ .

**Solution:**

$$\begin{aligned} \text{P.I} &= \frac{1}{D^2 + 7D + 14} (8e^{-x}) \\ &= 8 \cdot \frac{1}{D^2 + 7D + 14} (e^{-x}) \quad [\text{Replace } D \text{ by } (-1)] \\ &= \frac{1}{(-1)^2 + 7(-1) + 14} (e^{-x}) \\ &= \frac{8e^{-x}}{8} = e^{-x} \end{aligned}$$

2. Find the P.I of the DE  $(D^2 - 2D - 3) y = e^{-x}$ .

**Solution:**

$$\begin{aligned} \text{P.I} &= \frac{1}{D^2 - 2D - 3} (e^{-x}) \\ &= \frac{xe^{-x}}{f'(-1)} \\ &= \frac{xe^{-x}}{-4} \\ &= -\frac{xe^{-x}}{4} \end{aligned} \quad \left[ \begin{array}{l} f(-1) = (-1)^2 - 2(-1) - 3 \\ \quad = 1 + 2 - 3 \\ \quad = 0 \\ f'(D) = 2D - 2 \\ f'(-1) = 2(-1) - 2 \\ \quad = -4 \end{array} \right.$$

3. Find the P.I of the D.E  $(D^2 - 2D + 1) y = e^x$ .

**Solution:**

$$\begin{aligned} \text{P.I} &= \frac{1}{D^2 - 2D + 1} e^x \\ &= \frac{x^2 e^x}{f''(1)} \\ &= \frac{x^2 e^x}{2} \end{aligned}$$

$$\left[ \begin{aligned} f(1) &= 1^2 - 2(1) + 1 = 0 \\ f'(D) &= 2D - 2 \\ f'(1) &= 2(1) - 2 = 0 \\ f''(D) &= 2 \\ f''(1) &= 2 \neq 0 \end{aligned} \right.$$

**PART – C**

1. Solve:  $(D^2 - 5D + 6)y = e^{4x}$

**Solution:**

D.E is  $(D^2 - 5D + 6)y = e^{4x}$

A.E is  $m^2 - 5m + 6 = 0$

i.e,  $(m - 2)(m - 3) = 0$

$\therefore m = 2 \text{ or } m = 3$  [real & distinct roots]

$\therefore \text{C.F} = y = Ae^{m_1x} + Be^{m_2x} = Ae^{2x} + Be^{3x} \dots\dots\dots(1)$

$$\begin{aligned} \text{PI} &= \frac{1}{D^2 - 5D + 6} (e^{4x}) && \text{[Replace D by 4]} \\ &= \frac{e^{4x}}{4^2 - 5 \cdot 4 + 6} = \frac{e^{4x}}{2} && \dots\dots\dots(2) \end{aligned}$$

From (1) &amp; (2),

Solution,  $y = \text{C.F} + \text{P.I}$

i.e,  $y = Ae^{2x} + Be^{3x} + \frac{e^{4x}}{2}$

2. Solve:  $(3D^2 + D - 14)y = 13e^{2x}$

**Solution:**

D.E is  $(3D^2 + D - 14)y = 13e^{2x}$

A.E is  $3m^2 + m - 14 = 0$

i.e,  $3m^2 + 7m - 6m - 14 = 0$

i.e,  $m(3m + 7) - 2(3m + 7) = 0$

i.e,  $(3m + 7)(m - 2) = 0$

$\therefore 3m + 7 = 0 \text{ or } m - 2 = 0$

$m = -\frac{7}{3} \text{ or } m = 2$  [real and distinct roots]

$$\begin{aligned} \therefore \text{C.F} &= Ae^{m_1x} + Be^{m_2x} \\ &= Ae^{-2/3x} + Be^{2x} \dots\dots\dots(1) \end{aligned}$$

$$\begin{aligned}
 \text{PI} &= \frac{1}{3D^2 + D - 14}(13e^{2x}) \\
 &= 13 \frac{1}{3D^2 + D - 14}(e^{2x}) \\
 &= 13x \frac{1}{6D + 1}(e^{2x}) \\
 &= 13x \frac{e^{2x}}{6(2) + 1} \\
 &= \frac{13xe^{2x}}{13} = xe^{2x} \quad \dots\dots\dots(2)
 \end{aligned}$$

$$\left[ \begin{aligned}
 f(x) &= 3.2^2 + 2 - 14 \\
 &= 12 + 2 - 14 = 0 \\
 f'(D) &= 6D + 1 \\
 f'(2) &= 6(2) + 1 = 13
 \end{aligned} \right.$$

From (1) & (2), solution is

$$y = \text{CF} + \text{PI} = Ae^{-\frac{7}{3}x} + Be^{2x} + xe^{2x}$$

3. Solve:  $(D^2 - 6D + 9)y = 2e^{-x} + e^{3x}$

**Solution:**

D.E is  $(D^2 - 6D + 9)y = 2e^{-x} + e^{3x}$

A.E is  $m^2 - 6m + 9 = 0$

I.E,  $(m - 3)^2 = 0$

∴  $m = 3, 3$  (real and equal roots)

∴ C.F =  $(Ax + B)e^{mx} = (Ax + B)e^{3x}$  .....(1)

$$\begin{aligned}
 \text{PI}_1 &= \frac{1}{D^2 - 6D + 9}(2e^{-x}) \\
 &= 2 \frac{1}{D^2 - 6D + 9}(e^{-x}) \\
 &= 2 \frac{1}{(-1)^2 - 6(-1) + 9}(e^{-x}) \\
 &= \frac{2e^{-x}}{16} = \frac{e^{-x}}{8} \quad \dots\dots\dots(2)
 \end{aligned}$$

$$\left[ \begin{aligned}
 7(-1) &= (-1)^2 - 6(-1) + 9 \\
 &= 1 + 6 + 9 \\
 &= 16 \neq 0
 \end{aligned} \right.$$

$$\begin{aligned}
 \text{PI}_2 &= \frac{1}{D^2 - 6D + 9}(e^{3x}) \\
 &= x \frac{1}{2D - 6}(e^{3x}) \\
 &= x \left( x \cdot \frac{1}{2} e^{3x} \right) = \frac{x^2}{2} e^{3x} \quad \dots\dots\dots (3)
 \end{aligned}$$

$$\left[ \begin{aligned}
 7(3) &= 3^2 - 6.3 + 9 = 0 \\
 f'(D) &= 2D - 6 \\
 f'(3) &= 2(3) - 6 = 0 \\
 f''(D) &= 2 \\
 f''(3) &= 2 \neq 0
 \end{aligned} \right.$$

From (1), (2) & (3), solution is

$$y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$$

$$\text{i.e, } y = (Ax + B)e^{3x} + \frac{e^{-x}}{8} + \frac{x^2 e^{3x}}{2}$$

### 5.3 Solution of second order differential equations of the

$$\text{form } a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = \sin mx \text{ or } \cos mx ,$$

where  $a, b, c$ , are constants

To find the P.I for the D.E

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = \sin mx \text{ or } \cos mx$$

Case (i):

$$\begin{aligned} \text{P.I} &= \frac{1}{aD^2 + bD + c} (\sin mx) \\ &= \frac{1}{f(D)} (\sin mx) \quad [\text{Replace } D^2 \text{ by } (-m^2) \text{ upto the last step}] \end{aligned}$$

Case (ii):

$$\begin{aligned} \text{P.I} &= \frac{1}{f(D)} (\sin mx) \\ &= x \cdot \frac{1}{f'(-m^2)} (\sin mx) \quad [\text{when } D^2 \text{ becomes zero by replacing } D^2 \text{ with } -m^2] \end{aligned}$$

where  $f'(-a^2) \neq 0$

The same procedure is to adopted for finding PI when RHS of DE is  $\cos mx$ .

### WORKED EXAMPLES

#### PART – A

1. Find the C.F for the D.E  $(D^2 + 9) y = \sin x$

**Solution:**

DE is  $(D^2 + 9) y = \sin x$

A.E is  $m^2 + 9 = 0$

i.e,  $m^2 = -9$

$m = \pm \sqrt{-9} = \pm 3i$

$= 0 \pm i3$  (imaginary roots)

C.F =  $e^{\alpha x} (A \cos \beta x + B \sin \beta x)$

$= e^{0x} (A \cos 3x + B \sin 3x)$

$= 1 (A \cos 3x + B \sin 3x)$

$\therefore$  C.F =  $A \cos 3x + B \sin 3x$

2. Find the C.F for the DE  $(D^2 - 10D + 25)y = \sin 5x$ .

**Solution:**

$$\text{DE is } (D^2 - 10D + 25)y = \sin 5x$$

$$\text{AE is } m^2 - 10m + 25 = 0$$

$$\text{i.e. } (m - 5)^2 = 0$$

$$\therefore m = 5, 5 \text{ (real and equal roots)}$$

$$\begin{aligned} \therefore \text{CF} &= (Ax + B) e^{mx} \\ &= (Ax + B) e^{5x} \end{aligned}$$

### PART – B

1. Find the PI for the DE  $(D^2 + 25) y = \cos x$ .

**Solution:**

$$\begin{aligned} \text{PI} &= \frac{1}{D^2 + 25}(\cos x) \\ &= \frac{1}{-1^2 + 25} \cos x \quad [\text{Replace } D^2 \text{ by } -1^2] \\ &= \frac{\cos x}{24} \end{aligned}$$

2. Find the PI for the DE  $(D^2 - 4) y = \sin 3x$ .

**Solution:**

$$\begin{aligned} \text{PI} &= \frac{1}{D^2 - 4}(\sin 3x) \\ &= \frac{1}{-3^2 - 4} \sin 3x \quad [\text{Replace } D^2 \text{ by } -3^2] \\ &= \frac{1}{-13} \sin 3x = -\frac{\sin 3x}{13} \end{aligned}$$

3. Find the PI for the DE  $(D^2 + 16) y = \cos 4x$ .

**Solution:**

$$\begin{aligned} \text{PI} &= \frac{1}{D^2 + 16}(\cos 4x) \\ &= x \cdot \frac{1}{2D}(\cos 4x) \\ &= \frac{x}{2} \int \cos 4x \, dx \\ &= \frac{x}{2} \cdot \frac{\sin 4x}{4} = \frac{x \sin 4x}{8} \end{aligned} \quad \left| \begin{array}{l} f(-4^2) = -4^2 + 16 \\ \quad \quad = -16 + 16 \\ \quad \quad = 0 \\ f'(D) = 2D \end{array} \right.$$

## PART – C

1. Solve:  $(D^2 + 3D + 2)y = \sin 2x$

**Solution:**

DE is  $(D^2 + 3D + 2)y = \sin 2x$

AE is  $m^2 + 3m + 2 = 0$

i.e,  $(m + 2)(m + 1) = 0$

$m = -2, -1$  (real & distinct roots)

C.F =  $Ae^{m_1x} + Be^{m_2x}$

$= Ae^{-2x} + Be^{-x}$  .....(1)

$$\begin{aligned} \text{PI} &= \frac{1}{D^2 + 3D + 2}(\sin 2x) \\ &= \frac{1}{-2^2 + 3D + 2}(\sin 2x) \quad [\text{Replace } D^2 \text{ by } -2^2] \\ &= \frac{1}{3D - 2} \\ &= \frac{3D + 2}{(3D - 3)(3D + 2)}(\sin 2x) \\ &= \frac{3D + 2}{9D^2 - 4}(\sin 2x) \\ &= \frac{3D + 2}{9(-2^2) - 4} \quad [D^2 \text{ by } (-2^2)] \\ &= \frac{3D + 2}{-40} \sin 2x \\ &= -\frac{1}{40}[3D(\sin 2x) + 2 \sin 2x] \\ &= -\frac{1}{40}[3 \cos 2x \cdot 2 + 2 \sin 2x] \\ &= -\frac{1}{40} \cdot 2(3 \cos 2x + \sin 2x) \\ &= -\frac{(3 \cos 2x + \sin 2x)}{20} \quad \text{.....(2)} \end{aligned}$$

From (1) & (2),

Solution is  $y = \text{CF} + \text{PI}$

$= Ae^{-2x} + Be^{-x} - \frac{1}{20}(2 \cos 2x + \sin 2x)$

2. Solve:  $(D^2 - 5D + 6)y = x \cos 3x$

A.E is  $m^2 - 5m + 6 = 0$

i.e  $(m - 2)(m - 3) = 0$

$m = 2, m = 3$

CF =  $Ae^{2x} + Be^{3x}$

$$\begin{aligned}
&= 2 \cdot \frac{1}{D^2 - 5D + 6} (\cos 3x) \\
&= 2 \cdot \frac{1}{-3^2 - 5D + 6} (\cos 3x) \quad [\text{Replace } D^2 \text{ by } -3^2] \\
&= 2 \cdot \frac{1}{-5D - 3} (\cos 3x) \\
&= -2 \cdot \frac{1}{5D + 3} (\cos 3x) \\
&= -2 \cdot \frac{(5D - 3)}{(5D + 3)(5D - 3)} (\cos 3x) = -2 \cdot \frac{(5D - 3)}{25D^2 - 9} (\cos 2x) \\
&= -2 \cdot \frac{5D - 3}{25(-3^2) - 9} (\cos 3x) \quad [\text{Replace } D^2 \text{ by } -3^2] \\
&= -2 \cdot \frac{5(-\sin 3x^3) - 3 \cos 3x}{[25(-9) - 9]} \\
&= -2 \cdot \frac{(-15 \sin 3x - 3 \cos 3x)}{-234} \\
&= -\frac{1}{117} (15 \sin 3x + 3 \cos 3x) \\
&= -\frac{1}{39} (5 \sin 3x + \cos 3x) \quad \dots\dots\dots(2)
\end{aligned}$$

From (1) & (2)

Solution is  $y = CF + PI$

$$= Ae^{2x} + Be^{3x} - \frac{1}{39}(5 \sin 3x + \cos 3x)$$

3. Solve:  $(D^2 + 16)y = \cos 4x$

**Solution:**

DE is  $(D^2 + 16)y = \cos 4x$

A.E is  $m^2 + 16 = 0$

i.e,  $m^2 = -16$

$$\therefore m^2 = \pm \sqrt{-16}$$

$$= \pm i4$$

$$= 0 \pm i4 \quad [\text{imaginary roots } \alpha = 0, \beta = 4]$$

C.F =  $e^{\alpha x} (A \cos \beta x + B \sin \beta x)$

$$= e^{0x} (A \cos 4x + B \sin 4x)$$

$$= A \cos 4x + B \sin 4x \quad \dots\dots\dots (1)$$

$$\begin{aligned}
 \text{PI} &= \frac{1}{D^2 + 16} (\cos 4x) \quad [D^2 \text{ cannot be replaced by } (-4^2)] \\
 &= x \cdot \left( \frac{1}{2D} \cos 4x \right) \\
 &= \frac{x}{2} \cdot \frac{1}{D} (\cos 4x) \\
 &= \frac{x}{2} \int \cos 4x \, dx \\
 &= \frac{x}{2} \frac{\sin 4x}{4} = \frac{x \sin 4x}{8} \dots\dots\dots(2)
 \end{aligned}$$

From (1) & (2),

Solution is  $y = \text{C.F} + \text{P.I}$

$$\text{i.e. } y = A \cos 4x + B \sin 4x + \frac{x \sin 4x}{8}$$

## EXERCISE

### PART – A

1. If 5 and  $-6$  are the roots of the auxillary equation of second order differential equation with RHS zero, write the solution.
2. If 3, 3 are the roots of the auxillary equation of second order differential equation with RHS zero, write the solution.
3. If  $5 \pm 7i$  roots of the auxillary equation of second order differential equation with RHS zero, write the solution.
4. Solve:  $(D^2 - 49)y = 0$
5. Solve:  $y'' + 12y' + 36y = 0$
6. Solve:  $y_2 + 25y = 0$
7. Find the CF of the DE  $(D^2 - 3D + 2)y = e^{-4x}$ .
8. Find the CF of the DE  $(D^2 - 7D + 13)y = 5 \sin 7x$ .
9. Find the CF of the DE  $(D - 3)^2 y = 2e^{3x} - 7 \cos 4x$ .
10. Find the CF of the DE  $(D^2 - 5D + 6)y = 5e^{2x} - 3 \cos 9x$ .

### PART – B

1. Solve:  $\frac{d^2 y}{dx^2} - \frac{dy}{dx} + y = 0$
2. Solve:  $y'' + 6y' + 5y = 0$
3. Solve:  $y_2 + y_1 = 0$
4. Solve:  $(D^2 - 12D + 36)y = 0$
5. Solve:  $(D^2 + 13D - 90)y = 0$
6. Find the PI of  $(D^2 + D + 4)y = 10e^{2x}$ .
7. Find the PI of  $(D^2 - 8D + 15)y = e^{3x}$ .
8. Find the PI of  $(D^2 - 4D + 4)y = 3e^{2x}$ .
9. Find the PI of  $(D^2 + 10)y = \sin 3x$ .

10. Find the PI of  $(D^2 - 2D - 8)y = 5 \sin x$ .

### PART - C

1. Solve:  $(D^2 + 36)y = 0$ , when  $y(0) = 2$  and  $y'(0) = 12$ .
2. Solve:  $\frac{d^2y}{dx^2} + y = 0$ , given  $\frac{dy}{dx} = 2$  &  $y = 1$  when  $x = 0$ .
3. Solve:  $y_2 - 2y_1 + 15y = 0$ , given  $\frac{dy}{dx} = 0$ ,  $\frac{d^2y}{dx^2} = 2$  when  $x = 0$ .
4. Solve:  $(D^2 - D - 20)y = 0$ , given  $y(0) = 5$  and  $y'(0) = -2$ .
5. Solve:  $(D^2 + 3D + 2)y = 0$  if  $y(0) = 1$  and  $y'(0) = 0$ .
6. Solve:  $(2D^2 + 5D + 2)y = e^{-\frac{1}{2}x}$ .
7. Solve:  $(D^2 + 14D + 49)y = e^{-7x} + 4$ .
8. Solve:  $(D^2 - 13D + 12)y = e^{-2x} + 5e^x$ .
9. Solve:  $(D^2 - 5D + 19)y = 2e^{5x} - 3e^{-4x}$ .
10. Solve:  $(5D^2 + D + 1)y = 4e^{-6x}$ .
11. Solve:  $(D^2 - 3D + 2)y = 3e^{-4x} + 4e^{2x}$ .
12. Solve:  $(D^2 + D + 1)y = 2e^x - 45$ .
13. Solve:  $(D^2 + 4)y = \sin 3x$
14. Solve:  $(D^2 + 3D + 2)y = \sin 2x$
15. Solve:  $(4D^2 - 12D + 9)y = 7 \cos 2x$
16. Solve:  $(D^2 + 2D + 1)y = \sin x + \cos 3x$
17. Solve:  $(D^2 + 1)y = e^{4x} - 3 \cos x$
18. Solve:  $(D^2 - 25)y = \sin 2x$
19. Solve:  $(D^2 + 9)y = \sin 3x$
20. Solve:  $(D^2 + 7D + 6)y = 2e^{-x} + 4 \cos 5x - 7$

### ANSWERS

#### PART - A

- (1)  $y = Ae^{5x} + Be^{-6x}$       (2)  $y = (Ax + B)e^{3x}$       (3)  $y = e^{5x}(A \cos 7x + B \sin 7x)$   
 (4)  $y = Ae^{7x} + Be^{-7x}$       (5)  $y = (Ax + B)e^{-6x}$       (6)  $y = A + Be^{-25x}$   
 (7) C.F. =  $Ae^x + Be^{2x}$       (8) C.F. =  $e^{\frac{7x}{2}} \left( A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x \right)$       (9) C.F. =  $(Ax + B)e^{3x}$   
 (10) C.F. =  $Ae^{2x} + Be^{2x}$

#### PART - B

- (1)  $y = e^{\frac{x}{2}} \left( A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x \right)$       (2)  $y = Ae^{-5x} + Be^{-x}$       (3)  $y = A + Be^x$   
 (4)  $y = (Ax + B)e^{6x}$       (5)  $y = Ae^{-18x} + Be^{5x}$       (6)  $e^{2x}$       (7)  $-\frac{xe^{3x}}{2}$   
 (8)  $\frac{3x^2e^{2x}}{2}$       (9)  $\sin 3x$       (10)  $\frac{1}{17}(2 \cos x - 9 \sin x)$

**PART – C**

- (1)  $y = 2 \cos 6x + 2 \sin 6x$       (2)  $y = \cos x + 2 \sin x$       (3)  $y = \frac{1}{20}e^{5x} + \frac{1}{12}e^{-3x}$
- (4)  $y = 2e^{5x} + 3e^{-4x}$       (5)  $y = 2e^{-x} + e^{-2x}$       (6)  $y = Ae^{-\frac{1}{2}x} + Be^{-2x} + \frac{x}{3}e^{-\frac{1}{2}x}$
- (7)  $y = (Ax + B)e^{-7x} + \frac{x^2}{2}e^{-7x} + \frac{4}{49}$       (8)  $y = Ae^x + Ae^{12x} + \frac{e^{-2x}}{42} - \frac{5xe^x}{11}$
- (9)  $y = e^{\frac{5x}{2}} \left( A \cos \frac{\sqrt{51}}{2}x + B \sin \frac{\sqrt{51}}{2}x \right) + \frac{2e^{5x}}{19} - \frac{3e^{-4x}}{55}$
- (10)  $y = e^{-\frac{x}{10}} \left( A \cos \frac{\sqrt{19}}{10}x + B \sin \frac{\sqrt{19}}{10}x \right) + \frac{4e^{-6x}}{175}$
- (11)  $y = Ae^x + Be^{2x} + \frac{e^{-4x}}{10} + 4xe^{2x}$
- (12)  $y = e^{-\frac{x}{2}} \left( A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x \right) + \frac{2e^x}{3} - 45$
- (13)  $y = A \cos 2x + B \sin 2x - \frac{\sin 3x}{5}$
- (14)  $y = Ae^{-x} + Be^{-2x} - \frac{1}{20}(3 \cos 2x + \sin 2x)$
- (15)  $y = (Ax + B)e^{\frac{3x}{2}} - \frac{1}{625}(49 \cos 2x + 16 \sin 2x)$
- (16)  $y = (Ax + B)e^{-x} - \frac{1}{2} \cos x - \frac{1}{100}(8 \cos 3x - 6 \sin 3x)$
- (17)  $y = A \cos x + B \sin x + \frac{e^{4x}}{17} - \frac{3x}{2} \sin x$
- (18)  $y = Ae^{5x} + Be^{-5x} - \frac{\sin 2x}{29}$
- (19)  $y = A \cos 3x + B \sin 3x - \frac{x \cos 3x}{6}$
- (20)  $y = Ae^{-x} + Be^{-6x} + \frac{2xe^{-x}}{5} + \frac{1}{1586}(35 \sin 5x - 19 \sin 3x) - \frac{7}{6}$

# ***NOTES***

# ***NOTES***